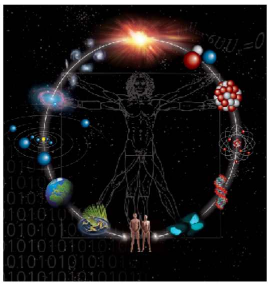

The Single Electron Transistor Renormalization Group analysis
The complete quantum theory of a complex system.

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Abstract.

The Single Electron Transistor (SET) is a metallic island coupled to two conducting reservoirs. In this masters research project a theory of physical observables is used to obtain the complete set of Renormalization Group equations for the SET. This theory defines new parameters that represent the quasi- charge and -conductance of the SET in the regimes of weak and strong coupling. The calculations that were made to derive the renormalization results are presented after an introduction into Single Electron Physics and the Coulomb blockade. The SET is found to have a robustly quantized quasi-charge. A complete phase diagram of the SET is presented. The fourth order correction to the action of the SET is found and analyzed. The effects of this correction to instanton solutions of the theory are studied. It is found that the correction to the instanton solutions are independent of the temperature.

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Contents

1. GENERAL SETUP.

The goal of this thesis is to show the fundamental features of the Single Electron Transistor(SET) and present next-order calculations that were done as a part of this masters project. First a very short summary of the project is given in Dutch. Then an introduction into Single Electron Physics is presented. Here a number of essential features of this research field such as the Coulomb blockade is introduced. Also some of the reasons for studying these systems are given and related topics are briefly discussed. In the following theory section a new theory that describes the physics of the SET is introduced and explained. This is done by following the steps in the paper in which it was published by Prof. A. M. M. Pruisken and Prof. I. Burmistrov [1]. First the theory is applied to a simpler system called the Isolated Island in section 4. Then in sections 5-8 this theory is used to describe the renormalization flow of the parameters that describe the SET. After this Theory section(4-8) calculations of the next-order correction to the theory are presented. First it is shown how to find the correction to the theory's action. Then its effect on the solutions of the bare action is studied. The thesis concludes with a recap of what was done and a conclusion that reflects the results and obtained knowledge.

2. DUTCH SUMMARY

This is a short summary of the project in Dutch.

2.1. De SET in het kort.

De single electron transistor, of SET, is een klein stukje geleidend materiaal waar electronen in en uit kunnen *tunnelen* vanaf twee kanten. Dit tunnelen van deeltjes wordt beschreven met de wetten van de kwantummechanica. Theoretisch natuurkundigen kunnen steeds meer aspecten van deze ingewikkelde systemen begrijpen en uitrekenen. In december 2009 lukte het de Amsterdamse proffesor Pruisken samen met een Russische collega voor het eerst dit SET systeem door te rekenen. Hieronder staat een schematische afbeelding van de SET:

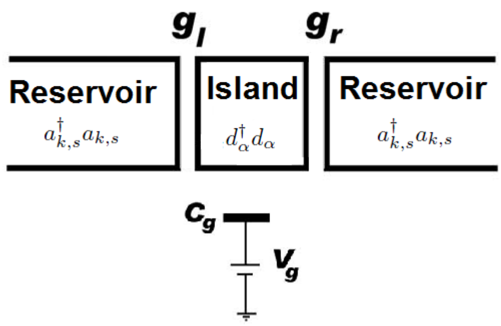

FIG. 1: De Single Electron Transistor.

Er staat een spanning over het systeem en hierdoor loopt er een stroom doorheen. De electronen die voor deze stroom zorgen moeten dus twee keer tunnelen. Bij deze stroom kan een geleidingscoëfficiënt geassociëerd worden en die noemt men g. Een van de belangrijkste vragen uit het onderzoek is of het aantal electronen op de SET aanwezig is voor alle waarden van g een exact te bepalen aantal is. Dit aantal electronen zorgt voor de lading van het eiland q. Als deze lading een geheel getal is dan noemt men dat gequantiseerd. Dit is een speciale eigenschap voor een groot systeem als de SET. Als dat ook voor grote waarden van g gebeurt dan is dat zelfs een zeldzaam en speciaal fenomeen. Het blijkt mogelijk om g en q net iets anders te definiëren als g' en q'. Uit deze nieuwe theorie volgt de quantisatie vanzelf. De SET vertoont dus voor alle waarden van g' een gequantiseerde quasi-lading q'.

Nadat er een introductie is gegeven in de natuurkunde die nodig is om deze systemen te beschrijven worden de hiervoor genoemde berekeningen uitgewerkt. Het resultaat zijn uiteindelijk enkele formules en een bijbehorend diagram die de SET helemaal beschrijven. Daarna wordt nog een correctie op deze resultaten uitgerekend. Het effect van deze correctie op de eerder gevonden resultaten wordt bestudeerd. Het blijkt dat deze correctie niet afhangt van de temperatuur van het systeem.

3. INTRODUCTION TO SINGLE ELECTRON PHYSICS.

In this section an introduction will be given to the physics of this master thesis. In this project the quantum mechanical properties of a system that is known as the Single Electron Transistor or SET is studied. The SET system will be introduced after an introduction is given into one of the most important phenomena in single electron physics, the Coulomb blockade. The SET is one of the simplest systems that show the Coulomb blockade.

3.1. Mesoscopic physics and quantum dots.

In this project the physical properties of a system called the Single Electron Transistor such as its conductance will be studied. Conductivity in materials is usually studied by combining the conductances of smaller regions of the material. In the second half of the last century it was discovered that quantum corrections may influence the conductance in certain system configurations. For instance if the size of a system gets of the order of the mean free path of the electrons. In this case the sense of conductivity is lost and the averaging to a total conductance can no longer be done. The local conduction properties of a system are then different then that of the systems total conductance. This can also happen when the temperature decreases to the point that quantum effect need to be taken into account. This temperature is then below the Thouless energy $E_T = \hbar/\tau_D$ where τ_D is the typical time for an electron in the system to diffuse. This energy scale defines the regime of mesoscopic physics being the regime in which quantum effect need to be taken into account. This regime has been studied both experimentally and theoretically intensively over the last decades [2–6]. In this project a system will be studied that is large in quantum standards but is still mesoscopic. It will be shown that the system that will be studied is very similar to quantum dots. A quantum dot is a incredibly small metallic area that is tailor made to host only a few conducting electrons. It can be seen as a point in which the number of conducting electrons can be controlled by an external gate. In the picture on the next page an example of such a system is given.

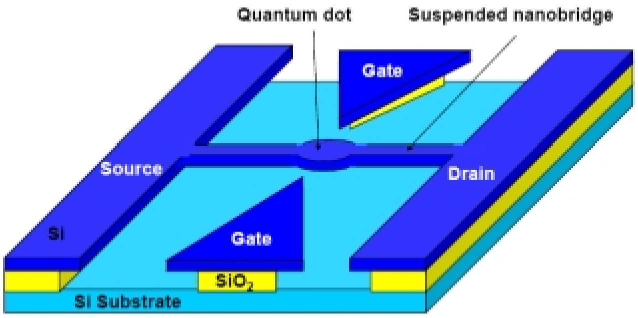

FIG. 2: Here the schematics of a typical quantum dot are shown. Electrons can go back and forth through the bridge and this flow is controlled by the gate.

It is important to note that no electrons can come from the gate. The contacts or brides can be formed in different ways. In quantum dot physics a very narrow piece of metallic material is frequently used but in this project tunneling contacts will be considered. In that case the dot or island is situated nearby one or two metallic reservoirs. The distance between the reservoirs and the island is close enough to allow conducting electrons to tunnel through the gap. Conducting electrons are electrons that are not bound to a single atom. They can freely move through a conducting material. There are also similar system configurations where the same physical phenomena comes into play[7]. For instance one can also consider superconducting materials separated by a normal conducting or insulating barrier, this is known as a Josephson junction. One of the most interesting features of these systems is the Coulomb blockade.

3.2. The Coulomb Blockade.

The Coulomb blockade is a result of the repellent interaction between electrons[8]. Considering a system that consists of two metallic regions separated by a small vacuum, it is possible for electrons in one side to tunnel to the other. This will certainly happen when a large bias voltage is applied across the system. Then the current that will flow through the system will be proportional to this bias voltage. The tunneling barrier now acts as a small resistance in the system. At very low bias voltages the capacitive nature of the tunnel junction also comes into play. When one electron has tunneled through the barrier a next electron will be repelled by the capacitance that has been placed over the barrier. This is why, at very low energies, only for certain values of the bias voltage an electron tunnels through. At these values the positive effect of the bias voltage is larger then the repulsive effect of the capacitance. A current can therefore only flow through the system at special values of the bias voltage. It must compensate the capacitance which is equal to an integer times the capacitance of a single electron; $U = \frac{e}{C_j}$. Here C_j is the junctions capacitance. This is called the Coulomb blockade.

In quantum dots the system looks somewhat different but most of the features are the same. Here the bridge in Figure 2 also acts as a capacitor at very low gate voltages. This is why only electrons can pass through the bridge when the gate voltage has overcome the capacitance effect of the junction. The quantum dot now receives electrons only at special and controlled values of the gate voltage. This is why the number of free electrons on the dot is always an integer. Or, equivalently the charge is always an integer times $-e$, the electron charge. In the figure below the current and charge of a typical quantum dot are given for a small increasing gate voltage.

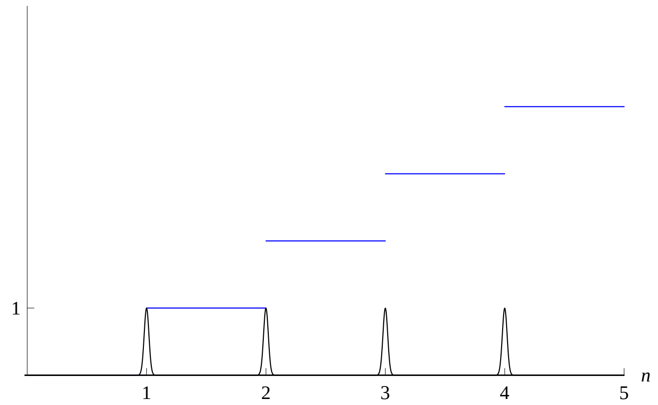

FIG. 3: Here the current through the system is plotted in black and the quantum dots charge in blue.

This makes it possible to create a transistor from the Coulomb blockade mechanism. The external gate potential acts as a switch that allows a current to pass through the barrier. By controlling the gate one can control the system. As a result the lowest energy state of a quantum dot depends on the number of free electrons on it. Also the flow of electrons to the dot is periodic and the potential build up is the same for each new electron. Since it is known from classical electrodynamics that the Coulomb energy of a zero dimensional system is proportional to the square of its charge one finds that the energy of the quantum dot as a function of the external charge looks as follows:

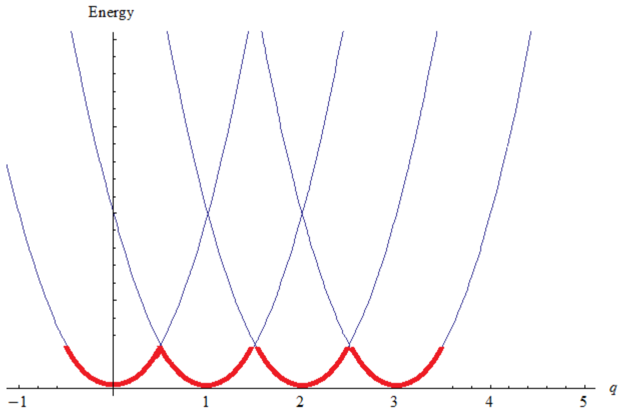

FIG. 4: The lowest energy state of the quantum dot for various numbers of electrons n on the quantum dot.

Each parabola curve is the energy potential associated with a certain number of island electrons n. When the electron number changes the system undergoes a quantum phase transition and goes from one potential curve over to the next. All these features will come back in this project numerous times and are caused by the Coulomb blockade. It has even been shown that the Coulomb blockade can also be studied without a barrier of junction. Yuli Nazarov showed that the same physics comes up when an arbitrary scatterer is studied in a metallic environment [9] and it is also an useful tool in studies of some types of granular metals[10]. This all shows that the physics that is studied in this project has wide applications in very different systems. Now the Coulomb blockade is discussed in a general way the Single Electron Transistor can be introduced.

3.3. The Single Electron Transistor System.

The Single Electron Transistor(SET) is basically a big quantum dot, a metallic island situated between two metallic reservoirs separated by tunneling junctions. However, there are some distinctions that can be made. A SET by its size is fundamentally different from a quantum dot. The SET island is big enough to be seen by the naked eye, as it can be a few millimeters big. This makes it a mesoscopic system, in the crossover region between the quantum world and macroscopic physics. Due to its macroscopic size it has been a challenge for many years to find out if the SET exhibits charge quantization and the Coulomb blockade like quantum dots. This is by no means clear because the number of energy levels in a SET is enormous and can be seen as a continuum. In a quantum dot it is possible to understand the system completely in terms of known energy levels but this is fundamentally impossible for the SET. This property makes it necessary to study electrons tunneling from reservoirs to the island while it is not possible to know what energy level they occupy.

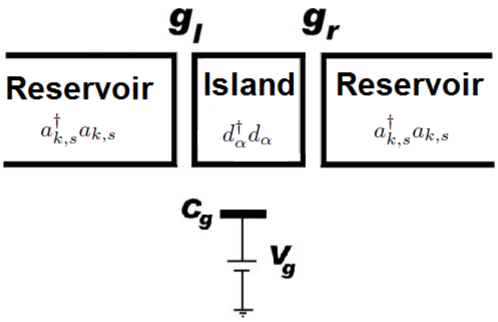

FIG. 5: This is a sketch of the SET.

In Figure 5 the quantum operators for the electrons in each section are given. They are not needed at this moment but one thing is important to note now. When for instance the average conductance through the system is calculated it will be necessary to sum over all possible energy states. This is one of the features of the SET that make it hard to describe. In the next figure a schematic of the SET that it is used in experimental studies is presented.

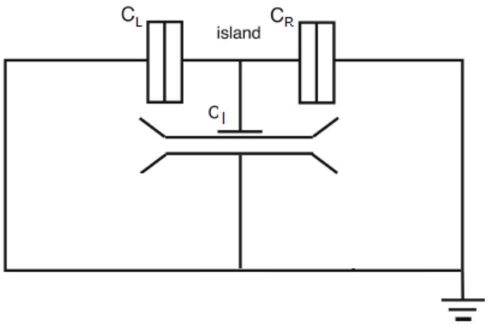

FIG. 6: This is a schematic of the SET that is used in experimental studies.

The island is simply the piece between the two tunneling barriers and the gate. The total capacitance of the system is the combined capacitance of the three and can be varied by varying the gate potential. Here the transistor nature of the SET becomes clear directly. By varying the gate one can control the number of electrons that tunnel to and from the island. Now a small sidestep to a simpler system that will be studied as well is necessary. This simpler system is needed to explain the questions of the quantum nature of the SET in more detail.

3.4. The Isolated Island.

Omitting the reservoirs for a moment, the system simplifies. A metallic island still in contact with an external gate is then left. This system is called the Isolated Island and has no experimental realization but is a thought experiment. If one thinks of the lowest energy state of the system it is no longer possible for a physical electron to come to the island. But the possible states of the island with different n still exist. That is why the plot of Figure 4 still describes the physics of the island correctly. This plot is associated to the quantum mechanical partition function of the isolated island:

$$Z = \sum_n e^{-\beta E_C (n-q)^2} \tag{1}$$

Here E_C is the Coulomb energy of the system, a positive constant. Also β is introduced as the inverse temperature. It is without doubt that this system has a quantized charge and this will be derived in the theory section in full detail. Although the island is big, its charge must be quantized due to the fact that electrons have a fixed charge and electrons come in integer amounts. This is why all the plots of the quantum dot can be associated to the isolated island as well. In Fig. 7 all functions related to the isolated island are combined.

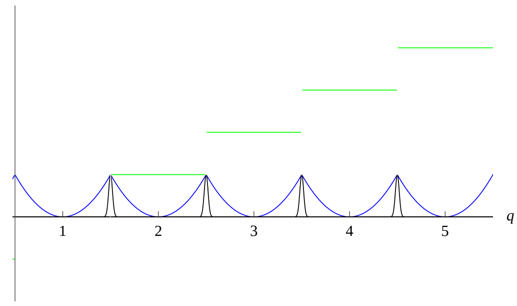

FIG. 7: In green the quantized charge is plotted and in black the current that only arises when the lowest energy(blue) states steps from one parabola to another.

At integer values of q the energy of the system is at its lowest point. At $q = n + 0.5$ an imaginary current can bring an extra electron to the system and the system energy is at its highest value.

3.5. The (un)quantized charge of the SET.

All these features disappear when a finite conductance is flowing through the system from one of the nearby reservoirs. When this is the case one needs to describe the system as a whole and must therefore take into account the infinite number of electrons in the reservoirs. Intuitively a quantized charge can no longer exist because of the reservoirs. In the 1980's Matveev has shown that the charge of the SET can no longer be quantized when a finite conductance is taken into account[11]. Combined with the years of effort physicists put into studies of the SET this result led to the general consensus that no quantization related to the SET was possible. Despite the believe that the SET's charge could no longer be quantized it stayed an interesting research topic for many theoretical physicists over the last decades. The literature concerned with the SET in different parameter regimes grew tremendously but no theory was found that could describe the system completely.

This is the starting point of this research project. The paper of Burmistrov and Pruisken from 2010[1] in which a complete theory for the SET was published will be followed. They showed that by studying the systems response to small violations of the boundary conditions of the SET fields they could find a complete theory. Their starting point is AES-theory, a theory that presents the mathematical structure of the SET Hamiltonian with a single abelian field ϕ. What the authors of the 2010 paper find is that one can define two new parameters by varying the boundary conditions of ϕ that described the SET system completely. The new parameters are very similar to the systems original charge and conductance variables q and g. By this new approach they find that the new variable q' is quantized. In the following theory section the steps that brought them to this remarkable result will be followed.

4. SET PHYSICS AND THE ISOLATED ISLAND.

4.1. Formal introduction to the SET.

In this section some of the content of the introduction will be repeated to formally define the Single Electron Transistor system. The system under consideration is the Single Electron Transistor or SET. It is a configuration of small pieces of metal in which the behavior of electrons is studied. In 2010 the system was for the first time completely described in a paper by Burmistrov and Pruisken [1]. In the next paragraphs the SET will be described by following the 2010 paper in detail. The SET's Renormalization Group equations and parameter flow will be derived. To analyze the SET one needs to develop a complete quantum mechanical description of the problem. The SET system looks as follows:

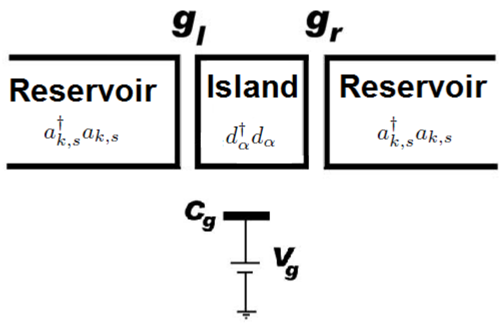

FIG. 8: This is a sketch of the SET.

Here $s = l, r$. One would like to calculate the charge on the island and the current that passes trough it due to tunneling from both sides. It is possible to consider a potential difference between the two reservoirs. This will result in a current going through the SET. Furthermore, an external gate potential can be considered that will externally tweak the SET's capacitance. By this means the systems response to chances by the external potential can be studied. The parameters of the system need to be defined. The operators associated with the island electrons are called $d^\dagger$ and d. On the reservoirs to the left and right of the island live electrons as well, in the following calculations the operators associated to those electrons are $a^\dagger_L, a_L, a^\dagger_R$ and a_R. These are all the initially given quanta. The electrons can tunnel from the reservoirs to the island and the other way around. To describe this behavior tunneling terms in our Hamiltonian description of the system are introduced. These tunneling terms will correspond to the left and right tunneling conductance g_L, g_R that are given in the picture.

The last thing missing in the global introduction is the interaction between the electrons. The interaction of the electrons in the reservoirs is assumed to be irrelevant for the behavior of the island electrons. This is why only the interaction on the island is taken into account. The Coulomb interaction for the island electrons has an associated energy $E_C = e^2/(2(C_L + C_R + C_I))$ where the C's correspond to the capacities on both reservoirs and the island. The Coulomb interaction between electrons is repulsive, when a charge difference exists between the island and the reservoirs the tunneling barrier obtains a capacitance. This is why one can say that the SET contains the Coulomb blockade effect; the electrons in the reservoirs that are pushed by a voltage difference to the island feel this repulsive capacitor effect. The Coulomb energy potential is given by $V_C = E_C(n-q)^2$. Here n is the total number of electrons on the island. The interactions on the island depends of course on the total external charge on the island. This charge is defined as $q = C_I V_I/e$. In Figure 9 the tunneling mechanism is drawn:

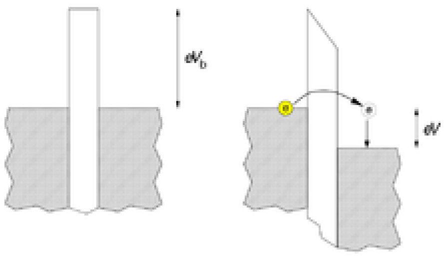

FIG. 9: Tunneling only happens if this is energetically favorable.

It is possible to tune C_I externally to vary the external charge and thereby the potential difference between island and banks. One can see that for special values of q the system allows a current to tunnel to the island. That is why the system behaves like a transistor.

The complete Hamiltonian of the SET can now be written down:

$$
\begin{aligned}
H_{SET1} = {} & (\sum_k \epsilon_{k,L} a^\dagger_{k,L} a_{k,L}) + (\sum_k \epsilon_{k,R} a^\dagger_{k,R} a_{k,R}) + (\sum_\alpha \epsilon_\alpha d^\dagger_\alpha d_\alpha) + E_C((\sum_\alpha d^\dagger_\alpha d_\alpha) - q)^2 \\
& + \sum_{k,\alpha} [t_{k,\alpha,L} a^\dagger_{k,L} d_\alpha + t_{k,\alpha,R} a^\dagger_{k,R} d_\alpha + t^*_{\alpha,k,L} d^\dagger_\alpha a_{k,L} + t^*_{\alpha,k,R} d^\dagger_\alpha a_{k,R}]
\end{aligned}
\tag{2}
$$

This can be written is a shorter way notated as:

$$
\begin{aligned}
H_{SET1} &= H_0 + H_C + H_T \\
&= \sum_{k,s=(L,R)} \epsilon_{k,s} a^{\dagger}_{k,s} a_{k,s} + \sum_{\alpha} \epsilon_{\alpha} d^{\dagger}_{\alpha} d_{\alpha} \\
&+ E_C (n-q)^2 \\
&+ \sum_{\substack{k,\alpha,\\ s=(L,R)}} t_{k,\alpha,s} a^{\dagger}_{k,s} d_{\alpha,s} + h.c.
\end{aligned}
\tag{3}
$$

Here $n = (\sum_{\alpha} d^{\dagger}_{\alpha} d_{\alpha})$, the electron number on the island. Also the $t_{k,\alpha,s}$'s are introduced. They stand for the tunneling matrix elements in the matrix that couples the reservoir states a_k to the island states d_{α}. The Hamiltonian is constructed by studying the system and recognizing the relevant energy contributions. Before 2010 is was not known that it was possible to completely understand this system. The Hamiltonian was known but physicists were not able to understand it well enough to calculate important physical quantities. In the next paragraphs this Hamiltonian will be transformed to describe the same physics with a simpler theory. Then the modified theory is used to calculate all the interesting physical quantities associated with the SET. Therefore a sidestep is necessary to introduce this new simpler theory. Later it will be proven that both theory's describe the same physics.

4.2. AES-theory and the Isolated Island.

In this section a theory is introduced that describes the SET by means of a path integral. First the SET is studied without the reservoirs to see the physics at work. This limited case is an instructive example for the physics to come. This case is called the Isolated Island for obvious reasons. One would like to find a fundamental quantum mechanical way to describe the external charge on the SET. It will be shown that the theory developed by Ambeogaokar, Eckern and Shön [12], called the AES-theory, is suitable for this cause. The problem with the Hamiltonian (3) is that the external charge is given in terms of classical quantities. The next lines show that is can be represented in another way. First one assumes that the physics on the isolated island can be described by a single abelian field depending on imaginary time, $\theta(\tau)$. Then one can write down the following quantum mechanical path integral to describe the system:

$$
\begin{aligned}
Z[q] &= \sum_{W=-\infty}^{\infty} e^{\pi 2iqW} Z_W \\
Z_W &= \int_{\theta(\beta)=\theta(0)+2\pi W} D\theta(\tau) e^{\int_0^\beta d\tau (\frac{-(\dot{\theta})^2}{(4E_C)})}
\end{aligned}
\tag{4}
$$

From the boundary condition $\theta(\beta) = \theta(0) + 2\pi W$ it follows that $\int_0^\beta d\tau\dot{\theta} = 2\pi W = \theta(\beta) - \theta(0)$. The classical equation of motion of the Coulomb term is $(\partial_\tau)^2\theta = 0$ so θ should be linear in τ. In physics one demands that two identical systems have the same equations of motion. This implies that $\theta(\tau) = 2\pi T\tau(W + \phi)$. Here W is the integer winding number that was introduced in the partition sum and ϕ was introduce as a background field that violates the boundary conditions of the theory. The winding number determines the topological configuration of the field θ. This is why one can say that ϕ carries fractional topological charge. Here θ is split in an integer winding number and the small variable ϕ which can only have values between - 0.5 and 0.5; or $\theta = W + \phi$. The partition function can then be written completely in terms of W and ϕ. To show the analogy to the SET's Hamiltonian ϕ is assumed to be zero for now.

$$
Z[q] = \sum_{W=-\infty}^{\infty} e^{\pi 2iqW} e^{\int_0^\beta d\tau (\frac{-(2\pi TW)^2}{(4E_C)} + iq2\pi TW)} \tag{5}
$$

Integration over τ results in an extra factor β and after cleaning up the argument of the exponential, it states: $2\pi Wiq - \frac{\pi^2 W^2}{\beta E_C}$. One then needs to use a Hubbard-Stratonovich transformation to rewrite it in such a way that the term that is quadratic in W is eliminated. The general form of a Hubbard-Stratonovich transformation is given by $e^{-AW^2} = \int_{-\infty}^{\infty} dy e^{-\frac{\alpha y^2}{2} + iyW}$. The classical equation of motions imply that $\alpha y_0 + iW = 0$ since $\partial_y(-\frac{\alpha y^2}{2} + iyW) = 0$. Using this to continue one finds:

$$
A = \frac{\pi^2}{\beta E_C} = \frac{1}{2\alpha} \tag{6}
$$

so

$$
\frac{\alpha}{2} = \frac{1}{4A} = \frac{\beta E_C}{4\pi^2} \tag{7}
$$

This can then be substituted into (5). In the second line the Poisson summation formula, $\sum_W e^{2\pi ixW} = \sum_n \frac{\delta(x-n)}{2\pi}$ is used to simplify the expression:

$$\begin{aligned} Z[q] &= \sum_{W=-\infty}^{\infty} e^{\pi 2iqW} e^{iyW-\beta E_C \frac{y^2}{\pi^2}} \\ &= \frac{1}{2\pi}\sum_n \delta(q+\frac{y}{2\pi}-n)e^{iyW-\beta E_C \frac{y^2}{\pi^2}} \\ &= \frac{1}{2\pi}\sum_n \delta(q+\frac{y}{2\pi}-n)e^{iyW-\beta E_C \frac{y^2}{\pi^2}} \\ &= \frac{1}{2\pi}\sum_n e^{-\beta E_C(n-q)^2} \end{aligned} \tag{8}$$

Notice that the second equation converges for $\beta E_C >> 1$ and the last equation does so for $\beta E_C << 1$. Therefore, it follows that the dual of the winding number W is the number of electrons on the island, n. This action describes the physics of the island completely in terms of winding numbers. This plot shows the potential for various values of n and in red the lowest energy state is drawn for all q:

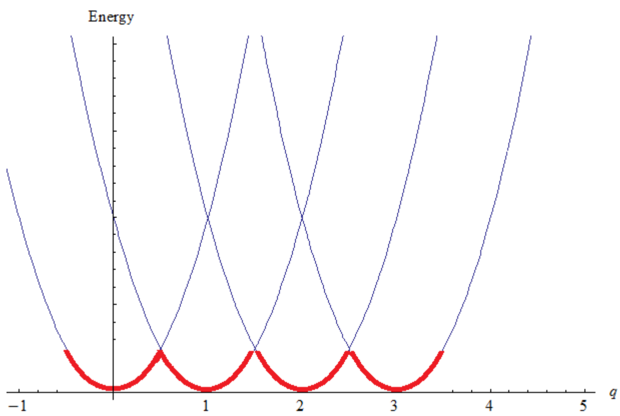

FIG. 10: This is the graph of Eq. (8).

Here ϕ is introduced as the perturbing background field that brings the system out of thermal equilibrium. For now it is zero but later ϕ will be used to perturb the theory that is described by the winding numbers. Therefore one can write the complete partition function in terms of ϕ and W as follows:

$$Z[q,\phi] = \sum_W e^{2\pi iq(W+\phi)-\frac{\pi^2}{\beta E_C}(W+\phi)^2} \tag{9}$$

To find out what q is in the new description of the theory one would like to split the ϕ and q dependence in $Z[q, \phi]$. Then one can for instance study the effect of the perturbations of ϕ to the charge on the island and other relations. To separate them a new action is introduced that is very similar to the original. If both actions are compared two new effective quantities are found, q' and E_C'. Both slightly differ from their original value by terms containing derivatives of $ln[Z(q)]$ with respect to q.

$$Z[q, \phi] = Z[q]e^{-S_{eff}} = Z[q]e^{-(-2\pi i[q' - \frac{\partial \ln Z[q]}{2\beta E_C \partial q}]\phi) - \frac{\pi^2}{\beta}[\frac{1}{E_C'(1+\frac{\partial^2 \ln Z[q]}{2\beta E_C \partial q^2})}]\phi^2 + O(\phi^3)} \tag{10}$$

This is in itself logically consistent and introduces the following two new variables:

$$q' = q + \frac{\partial \ln Z[q]}{2\beta E_C \partial q} \tag{11}$$

$$\frac{1}{E_C'} = \frac{1}{E_C}(1 + \frac{\partial^2 \ln Z[q]}{2\beta E_C \partial q^2}) \tag{12}$$

Higher power terms of the background field are considered to be irrelevant because they decrease exponentially with the temperature. Furthermore, if one would do the same thing not in terms of winding numbers but in the representation with the number of particles n one finds:

$$Z[q, \phi] = Z[q]e^{-S_{eff}} = Z[q]e^{-2\pi i\langle n\rangle\phi + 2\pi(\langle n^2\rangle - \langle n\rangle^2)\phi^2 - \mathcal{O}(\phi)^2} \tag{13}$$

Comparing this with the winding number representation it is found that q' is the average number of particles n and the effective Coulomb interaction is related to the variance of n.

$$q' = < n > \quad , \quad \frac{1}{\beta E_C'} = 8(< n^2 > - < n >^2) \tag{14}$$

To isolate q and E_C Eq. (8) is split into a periodic and non-periodic part:

$$\begin{aligned} Z[q, \phi] &= e^{2\pi i\phi k(q)} Z[\frac{\theta(q)}{2\pi}, \phi] \\ Z[\frac{\theta(q)}{2\pi}, \phi] &= \sum_{n'} e^{2\pi i\phi n' - \beta E_C(n' - \frac{\theta(q)}{2\pi})^2} \end{aligned} \tag{15}$$

this is the first preliminary result which will be analyzed later. First the mathematical structure of the SET's path integral will be studied and the influence of the background field ϕ will be analyzed. This knowledge is needed to study the SET's path integral for both the isolated and the connected case. From these path integrals it will be possible to derive the Renormalization Group equations of the SET in terms of the conductance and this quasi-charge q'.

4.3. Mathematical Structure.

In this paragraph the mathematical structure of the problem will be studied. The SET has a non-trivial topology and it is therefore useful to study the structure of our formulas. By this way one will find possible calculation schemes which will simplify the calculations to come. This analysis is done for the complete SET including reservoirs. First the partition function that was found including the tunneling action is presented:

$$Z_W = -\int_{\phi(\beta)=\phi(0)+2\pi W} \mathcal{D}\phi(\tau)e^{-S_t(\phi)-S_C(\phi)} \tag{16}$$

Here S_t is the part of the action that describes the tunneling from the reservoirs to the island. This term is proportional to the conductance. For this reason it was absent when the isolated island was considered. When there are reservoirs in the neighborhood of the SET, electrons will tunnel as described by the following action:

$$S_t = \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{1-2})e^{i\phi(\tau_1)-i\phi(\tau_2)} \tag{17}$$

A detailed description of g will be presented in the section on tunneling channels. Essentially it is the coupling constant of the tunneling action. The tunneling actions physics is described by a kernel α , given in both time and frequency representation below.

$$\alpha(\tau) = -T^2\text{cosec}^2(\pi T\tau) = \frac{T}{\pi}\sum_{\omega_n} \mid \omega_2 \mid e^{i\omega_n\tau} \tag{18}$$

Here ω_n is a bosonic frequency $2\pi Tn$. The kernel has a period of β, the inverse temperature. Its argument is the difference between two imaginary times $\tau_{1-2} = \tau_1 - \tau_2$. The Coulomb interaction part of the action is the same as in the isolated island case:

$$S_C(\Phi) = \frac{1}{E_C}\int_0^\beta d\tau\dot{\Phi}^2 \tag{19}$$

The action (16) can be represented more elegantly by a $\mathbb{O}(2)$ field variable:

$$Q(\tau) = \begin{pmatrix} \cos\Phi & \sin\Phi \\ \sin\Phi & -\cos\Phi \end{pmatrix} \tag{20}$$

When Q is multiplied by itself one can easily see that it becomes unity. The total action will now only depend on Q and when it is integrated over with periodic boundary conditions it looks like this:

$$S(Q) = \int_0^\beta d\tau_1 d\tau_2 \gamma(\tau_{1-2})\text{Tr}[Q(\tau_1)Q(\tau_2)] - \frac{q}{2}\int_0^\beta d\tau\sigma_y Q\partial_\tau Q \tag{21}$$

with σ_y the Pauli matrix in the y direction. Here the new kernel γ in frequency representation is given by:

$$\gamma(i\omega_n) = \frac{g}{4\pi} \mid \omega_n \mid + \frac{1}{8\pi}\omega_n^2 \tag{22}$$

One can still recognize a part proportional to g and a part that is quadratic in the frequency. If one takes a closer look at the term which is proportional to the charge q it is found that the theory only depends on q modulo 2π. When q is split into a integer and periodic part one can see that the step between integers can only be made out of thermal equilibrium. In the picture below both parts of $q = k(q) + \theta(q)$ can be seen .

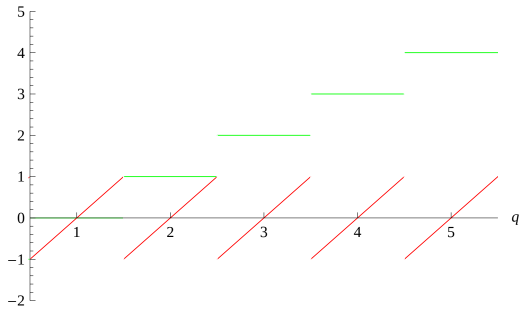

FIG. 11: Here $k(q)$ is green and $\theta(q)$ is plotted in red.

To extract $k(q)$ from the theory one must introduce fields that take the SET out of thermal equilibrium. The total winding number must be changed to step from one value of k to another. Because the theory is thus strongly dependent on the winding number people say that W labels different topological sectors of the theory, since the structure of the fields is different for different W.

4.4. Instanton solution of the path integral.

For each value of W the path integral (16) has a stable minimum. Each value of W defines a topological sector. It defines a certain class of field configurations. One can therefore also say that each topological sector has a stable solution. These solutions were first found in the context of condensed matter physics by Korshunov in 1987[13]. The name of these solutions was introduced by 't Hooft [14]. These solutions are very similar to Yang-Mills instantons [15] and can be represented by the following formula:

$$e^{i\Phi_W(\tau)} = \prod_{a=1}^{|W|} \frac{1 - z(\tau) z_a}{z(\tau) - z_a^*} \qquad \text{with} \qquad z(\tau) = e^{-i2\pi T\tau} \tag{23}$$

In the limit of large conductance these solutions dominate the physics of the system. The SET can be though of as a dilute gas of single instantons($|z_a|^2 > 1$) and their conjugate anti-instantons ($|z_a|^2 < 1$) . The calculations related with this instanton solutions will be done at the end of this project. Here some of its phenomenology is discussed. Also the RG equations of the instantons are included for later comparison. The instantons are pulse events in imaginary time. When an instanton event takes place W changes by an integer value. This can be made more clear by the corresponding plot:

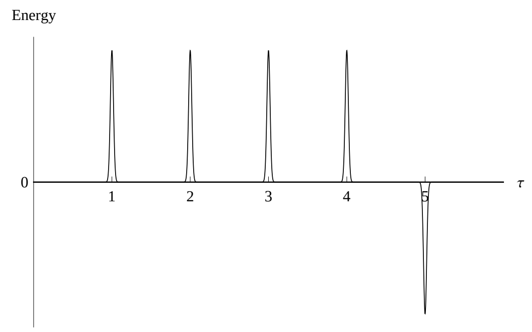

FIG. 12: This a graph of four instanton events followed by a anti-instanton event.

At these moments it is possible for the average charge on the island to increase of decrease. This is shown in Figure 13, at each instanton event the green charge line increases.

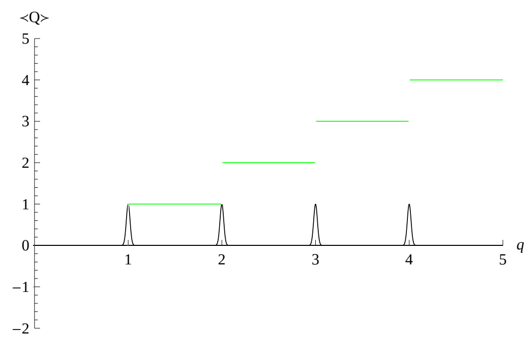

FIG. 13: This is a graph of instanton events(black). Also the average charge $\langle Q \rangle$ is plotted in green.

The radiative corrections that can be found from ordinary perturbation theory [16] are presented here because they will later be included in the renormalization equations of the total SET. If one defines $\lambda = \beta(1- \mid z_1 \mid^2)$ as the scale size of a potential pulse $i\Phi_{W\pm1}(\tau)$ the following scaling equations of the conductance and Coulomb Energy can be extracted:

$$\beta_g = \frac{dg}{d\ln(\lambda)} = -2 - \frac{4}{g(\lambda)}, \qquad \beta_{E_C} = \frac{d\ln E_C(\lambda)}{d\ln(\lambda)} = -\frac{2}{g(\lambda)} \tag{24}$$

Both contain the energy cutoff dependent conductance $g(\lambda) = g - 2\ln(\lambda\Lambda)$. This renormalization of the instanton solutions must be taken into account when the SET is considered as a whole. Now the analysis of the SET is continued and study of the isolated island is finished.

4.5. Physical observables.

The RG equations presented above are nice results but only show the non critical phases of the SET and the isolated island. The Renormalization Group is regularly used to show the critical points of a theory and differentiate between different phases of state. This is done by absorbing ultraviolet or infrared divergences of the theory into energy depended coupling constants. This then allows for the calculation of critical exponents of a theory like the correlation length and other observables. There are different ways to treat these divergences and one usually chooses one that sooths the system best.

However, the SET is a more challenging system since the instantons do not describe the physics at low coupling. The low energy dynamics are only weakly influenced by the non-critical instanton sector. At low energy's infrared divergences are encountered. Here the Coulomb blockade and possible critical points change the renormalization behavior. These infrared infinities must be dealt with to obtain a good understanding of the field θ that describes the physics at very low energies. This has been proven to be an enormous challenge represented by the extensive list of unsuccessful attempts to capture this parameter regime; for a complete list of references see [1]. A new and revolutionary technique that was introduced in the theory of the quantum Hall effect can shed light on these problems. A theory of physical observables that describes the system can be developed by studying the effect of a varying background fields ϕ. If one studies the effect that infinitesimally small changing boundary conditions made by ϕ have on the field θ one can find the renormalization behavior of the SET in this regime. The relatively simple action of the SET that still exhibits this complicated physical behavior makes this system an ideal playground to see these mechanisms at work. To see this physics at work first the analysis of the isolated island is completed. Then the same mechanisms are used to analyze the much more complicated SET.

It was found that in the rewritten completely quantum mechanical action the integer n represented the number of electrons on the island. This was found by making an expansion in the field ϕ that varied from an integer winding number W. Below the equation with separated q and ϕ, Eq. (15), is rewritten for convenience:

$$\begin{aligned} Z[q,\phi] &= e^{2\pi i\phi k(q)} Z[\frac{\theta(q)}{2\pi},\phi] \\ Z[\frac{\theta(q)}{2\pi},\phi] &= \sum_{n'} e^{2\pi i\phi n' - \beta E_C (n' - \frac{\theta(q)}{2\pi})^2} \end{aligned} \tag{25}$$

With the knowledge obtained in the last paragraphs it is possible to analyze this result. If θ is exactly equal to $\pm\pi$ the system will loose its energy gap and electrons will be able to tunnel through the barrier. In the zero temperature limit the second part of the action dominates and it is for that reason independent of ϕ. This is what is expected since at zero temperature there should not be any fluctuations in the background field. At half integer values of q' the number of electrons on the island changes which is thereby related to $k(q)$.

$$q' = \langle n \rangle = k(q) \tag{26}$$

The Renormalization Group behavior of the isolated island can now be studied.

4.6. Renormalization of the Isolated Island.

At zero temperature the island is ϕ-independent. Also there is a restriction for $\theta(q)$ between $\pm\pi$ by definition. The island develops an energy gap Δ_0 in this section only when $\theta(q) = \pm\pi$. Then the potential is integer valued and the electrons can freely tunnel in and out of the SET. One can see that indeed q' is quantized and that the Coulomb potential diverges periodically at half integer values of q. At zero temperature the thermodynamic potential is given by:

$$\Omega = E_C(\frac{\theta(q)}{2\pi})^2 \tag{27}$$

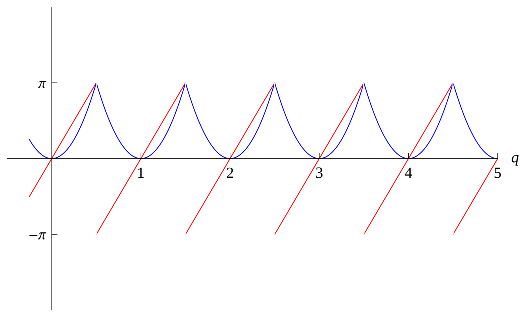

FIG. 14: In red the field θ is plotted and in blue Ω.

In the figure above θ and $\Omega \sim \theta^2$ are plotted. By expanding the equations in powers of ϕ one can describe the renormalization at finite temperature. One can see that the sum of equation (15) is mainly dominated by 0 and ± 1 which allows one to write down an effective action for these values of n.

$$S^0_{eff} = -i\theta^{'}\phi - \frac{\pi^2\phi^2}{\beta E^{'}_C} + \mathcal{O}(\phi^3) \tag{28}$$

With this action it is possible to calculate the thermodynamic potential and the physical observables. When this effective action is inserted in the partition sum the energy gap naturally occurs. Define the energy gap as: $\Delta_0 = 1 - |\frac{\theta(q))}{\pi}|$ to find:

$$\begin{aligned}
\Omega(q) &= -\ln Z[\frac{\theta(q)}{2\pi}] \\
&= \frac{\beta E_C(1-\Delta_0)^2}{4} - \ln[1 + e^{-\beta E_C \Delta_0}] \\
\theta^{'} &= \theta(q) - \frac{\pi\partial\Omega(q)}{E_C\partial q} = \pm\frac{2\pi}{1+e^{-\beta E_C\Delta_0}} \\
\frac{1}{\beta E^{'}_C} &= \frac{1}{\beta E_C}(1 - \frac{\partial^2\Omega(q)}{2E_C\partial q^2}) = |\frac{\theta^{'}}{2\pi}|(1 - |\frac{\theta^{'}}{2\pi}|)
\end{aligned} \tag{29}$$

These quantities can be put in differential form to show the Renormalization flow:

$$\beta_\theta(\theta^{'}) = \frac{d\theta^{'}}{d\ln\beta} = \frac{\theta^{'}}{2\pi}(2\pi - |\theta^{'}|)\ln[\frac{|\theta^{'}|}{2\pi - |\theta^{'}|}] \tag{30}$$

Figure 15 shows a plot of this RG β-function.

The periodicity in θ can be recognized. The Coulomb energy just depends on θ and not of ϕ. This means that the results above gives the first RG results of this project. It describes the isolated island at finite temperatures. These equations are used to study the fixed points and phases of state of the isolated island. One can see that the function crosses zero at 0 and $\pm\pi$. These points

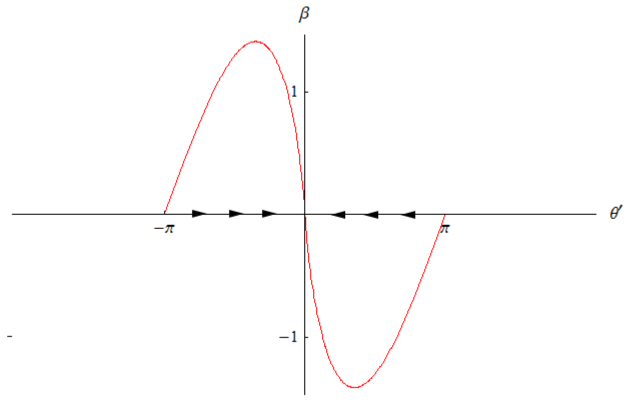

FIG. 15: This is a graph of the Renormalization flow of the Isolated Island around $q = 0$.

are called the strong coupling fixed points. The name strong coupling will be introduced later for small values of the conductance parameter g. The points differ in nature. The point at $\theta^{'} = 0$ is a stable fixed point where the β-function goes as follows:

$$\beta_\theta(\theta^{'}) = \theta^{'} \ln \mid \theta^{'} \mid \tag{31}$$

The corrections to the charge on the island near this point are thus exponentially small in β. From the definition of q it follows that $q^{'} = k(q) \pm e^{-\beta E_C \Delta_0}$. The charge on the island is therefore robustly quantized and the fluctuations go to zero with the temperature. At $T = 0$ the isolated island's charge is quantized. Here $\langle Q \rangle(q)$ is given for two different temperatures:

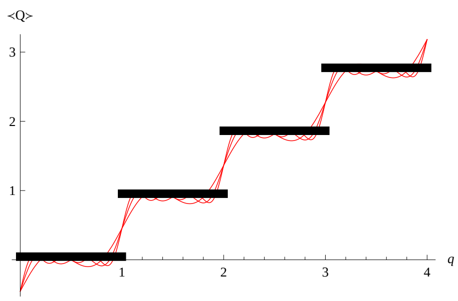

FIG. 16: In this graph the quantized charge is plotted for $T = 0$(black) and $T\rangle 0$(red).

The second fixed points are critical. Because of the periodicity in q one can better speak of a line of critical points all separated by 2π. The plots and equations in this paragraph are all focused on the behavior between $-\pi$ and π. Near these critical points the field is linear which implies that a first order transition happens here. This is the final result that can be obtained from the isolated island.

Now the full Single Electron Transistor will be studied. Here electrons can tunnel in and out of the island from nearby reservoirs. Before it is possible to start analyzing the complete SET the tunneling behavior of the electrons and their quantum description will need to be discussed. When the tunneling is understood the conductance parameter g can be constructed.

4.7. Tunneling channels.

In this paragraph a review of the tunneling terms that are part of the SET's Hamiltonian will be given. It will be shown how they can be related to the conductance that flows through the SET. First the tunneling Hamiltonian of the SET that was given in Eq. (3) is repeated:

$$\mathcal{H}^{(s)}_{tun} = \sum_{\substack{k,\alpha,\\ s=(L,R)}} t_{k,\alpha,s} a^{\dagger}_{k,s} d_{\alpha,s} + h.c. \tag{32}$$

The elements $t_{\alpha k}$ contain the amplitudes for tunneling between the dot states d_α and the reservoir states $a^{\dagger}_{k}$. When this is turned into path integral language one finds the following action:

$$S_{tun} = -\text{Tr Log} \begin{bmatrix} G^{-1}_{\alpha} & te^{i\phi} \\ e^{-i\phi}t^{\dagger} & G^{-1}_{a} \end{bmatrix} \tag{33}$$

$$S_{tun} = -\text{Tr Log } (\begin{bmatrix} G^{-1}_{\alpha,n} & 0 \\ 0 & G^{-1}_{a,n+m} \end{bmatrix} + \begin{bmatrix} 0 & te^{i\phi}_{m} \\ e^{-i\phi}_{-m}t^{\dagger} & 0 \end{bmatrix}) \tag{34}$$

This action will be studied more closely in the next sections. For now notice that the tunneling dependent part of the action can be extracted. The matrix structure can be split in two parts. The tunneling part can be extracted from the part that contains the inverse Greens functions. These Green functions will be introduced later. Then some matrix elements are constructed from the tunneling amplitudes that can be translated into conductances that flow from one of the reservoirs to the island and vise versa:

$$\hat{g}^{(s)}_{kk'} = 4\pi^2 \left(\delta(\epsilon^{(s)}_{k}) \delta(\epsilon^{(s)}_{k'}) \right)^{0.5} \sum_{\alpha} t^{(s)}_{k\alpha} \delta(\epsilon_{\alpha}) t^{(s)\dagger}_{\alpha k'} \tag{35}$$

$$\check{g}^{(s)}_{\alpha\alpha'} = 4\pi^2 \left(\delta(\epsilon^{(s)}_{\alpha}) \delta(\epsilon^{(s)}_{\alpha'}) \right)^{0.5} \sum_{k} t^{(s)}_{\alpha k} \delta(\epsilon_{k}) t^{(s)\dagger}_{k\alpha'} \tag{36}$$

With these it is possible to build the total conductance g.

$$\begin{aligned} g &= g^l + g^r = \sum_k g^l_{kk} + \sum_k g^r_{kk} \\ &= 4\pi^2 \left(\delta(\epsilon^l_k)\delta(\epsilon^l_{k'})\right)^{0.5} \sum_\alpha t^l_{k\alpha}\delta(\epsilon_\alpha)t^{l\dagger}_{\alpha k'} + 4\pi^2 \left(\delta(\epsilon^r_k)\delta(\epsilon^r_{k'})\right)^{0.5} \sum_\alpha t^r_{k\alpha}\delta(\epsilon_\alpha)t^{r\dagger}_{\alpha k'} \end{aligned} \tag{37}$$

It is still possible to treat the left and right contribution separately. The term under the sum determines the direction of flow. The tunneling elements are extremely important quantum operators in the analysis of the SET. The delta functions stand for the mean level spacing on the island and the reservoirs. An electron can only tunnel in a vacant quantum state. That process is governed by the expression above and it does also depend on the external gate potential that effects these quantum states. For each nonzero eigenvalue of the matrices above there exists a transport channel. An electron that reaches the boundary of the reservoir can tunnel to the island if the eigenvalue of the tunneling matrix at that point in time is nonzero for his energy level. The total number of these channels is given by:

$$N^{(s)}_{ch} = \frac{(\sum_k g_{kk})^2}{\sum_{k,k'} g_{k,k'}g_{k',k}} = \frac{(\sum_\alpha g_{\alpha\alpha})^2}{\sum_{\alpha,\alpha'} g_{\alpha,\alpha'}g_{\alpha',\alpha}} \tag{38}$$

One can also determine the conductance per channel:

$$g^{(s)}_{ch} = \frac{\sum_{k,k'} g_{k,k'}g_{k',k}}{\sum_k (g_{kk})} = \frac{\sum_{\alpha,\alpha'} g_{\alpha,\alpha'}g_{\alpha',\alpha}}{\sum_\alpha (g_{\alpha\alpha})} \tag{39}$$

In this project only the case where $g_{ch} \ll 1$ is considered. This is natural for the quantum domain. For the SET system the number of channels is always large so the total conductance g can still be much bigger then 1. The conductance at each site of the island is then given by:

$$g^{(s)} = g^{(s)}_{ch} N^{(s)}_{ch} \tag{40}$$

It is also assumed that the mean level spacing is negligibly small, $\delta \ll 1$ and the Coulomb energy is much bigger that the spacing $E_C \gg \delta$. With these restrictions it is possible to ignore the exchange energy. This does not limit the investigation in any way since all these limits are natural for the SET system. Because the inverse of the numbers of channels will be used it is presented here as well:

$$
\begin{aligned}
\frac{1}{N_{ch}} &= \frac{\sum_{k,k^{‘}} g_{k,k^{‘}} g_{k^{‘},k}}{(\sum_k g_{kk})^2} \\
&= \frac{\sum_{\alpha,\alpha^{‘}} g_{\alpha,\alpha^{‘}} g_{\alpha^{‘},\alpha}}{(\sum_\alpha g_{\alpha,\alpha})^2} \\
&= \frac{\sum_{kk'}[(\delta(\epsilon_k^{(s)})\delta(\epsilon_{k'}^{(s)})))^{0.5}\sum_\beta t_{k\beta}^{(s)}\delta(\epsilon_\beta)t_{\beta k'}^{(s)\dagger}] \times [\delta(\epsilon_k^{(s)})\delta(\epsilon_{k'}^{(s)})^{0.5}\sum_\gamma t_{k'\gamma}^{(s)}\delta(\epsilon_\gamma)t_{\gamma k}^{(s)\dagger}]}{(\sum_k(\delta(\epsilon_k^{(s)})\delta(\epsilon_k^{(s)}))^{0.5}\sum_\alpha t_{k\alpha}^{(s)}\delta(\epsilon_\alpha)t_{\alpha k}^{(s)\dagger})^2} \\
&= \frac{\sum_{\alpha\alpha'}([(\delta(\epsilon_\alpha^{(s)})\delta(\epsilon_{\alpha'}^{(s)}))^{0.5}\sum_\beta t_{\alpha\beta}^{(s)}\delta(\epsilon_\beta)t_{\beta\alpha'}^{(s)\dagger}][\delta(\epsilon_\alpha^{(s)})\delta(\epsilon_{\alpha'}^{(s)})))^{0.5}\sum_\gamma t_{\alpha'\gamma}^{(s)}\delta(\epsilon_\gamma)t_{\gamma\alpha}^{(s)\dagger}]}{(\sum_\alpha(\delta(\epsilon_\alpha^{(s)})\delta(\epsilon_\alpha^{(s)}))^{0.5}\sum_\alpha t_{\alpha\alpha}^{(s)}\delta(\epsilon_\alpha)t_{\alpha\alpha}^{(s)\dagger})^2}
\end{aligned}
\tag{41}
$$

Notice that A. Altland in his book and texts often introduces a different conductance that is defined after an integration over one of the field variables [10, 17]. It is then possible to combine the tunneling elements and continue with the following conductance:

$$g_t = 4\pi^2\rho_\alpha\rho_a|t|^2$$

This will only be used in section 9 where new calculations are presented because it is very useful there. Now that the relevant quantum operators are found it is possible to include the tunneling physics of the SET. Now that these quantities are introduced the connected island can be studied and its phase-diagram can be derived.

5. THE SINGLE ELECTRON TRANSISTOR.

In this section the SET will be studied with finite values of g.

5.1. The complete action S_{SET}.

In this paragraph the action of the isolated island will be modified to include the tunneling from the reservoirs. To do this the kernel has to become time dependent. When the SET is studied at finite conductance one must still allow for the field ϕ to perturb the complete system. Therefore ϕ must also influence the tunneling phenomena. This requires a slight modification of the kernel. First the argument of the kernel is altered to include the term $2\pi T\phi$. Before the kernel was Eq. (18) and this now becomes:

$$\alpha_\phi(\tau_{12}) = \frac{T}{\pi}\sum_n \mid \omega_n + 2\pi T\phi \mid e^{-i(\omega_n+2\pi T\phi)\tau_{12}} \tag{42}$$

If one includes the time evolution as well the following alteration must be made:

$$\alpha(\tau_{12}) \to e^{i(2\pi T\phi)\tau_1}\alpha_\phi(\tau_{12})e^{-i(2\pi T\phi)\tau_2} = \alpha(\tau_{12}) + 2T^2 \mid \phi \mid -2iT^2\phi\cot(\pi T\tau_{12}) \tag{43}$$

If one splits the exponentials like in the equation above the background field ϕ is extracted and the original kernel is obtained with corrections proportional to ϕ. It is also necessary to make the background field time depended for this investigation. These demands are met in the field $\Phi_0(\tau) = \tau(\omega_n + 2\pi T\phi)$. With this modified field the action becomes:

$$\begin{aligned} S(\Phi+\Phi_0) &= \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 e^{i\Phi(\tau_1)-i\Phi(\tau_2)+i\Phi_0(\tau_1)-i\Phi_0(\tau_2)}\alpha_\phi(\tau_{12}) \\ &= \frac{g}{4\pi}\int_0^\beta d\tau_1 d\tau_2 e^{i\Phi(\tau_1)-i\Phi(\tau_2)}T\sum_n e^{-i\omega_n\tau_{12}} \mid \omega_n + \dot{\Phi}_0 \mid \end{aligned} \tag{44}$$

This action no longer only describes the isolated island case but also includes the tunneling electrons. This gives the following total action:

$$S_{SET} = \int_0^\beta d\tau(\frac{-(\dot{\theta})^2}{(4E_C)} + \frac{g}{4\pi}\int_0^\beta d\tau_1 d\tau_2 e^{i\Phi(\tau_1)-i\Phi(\tau_2)}T\sum_n e^{-i\omega_n\tau_{12}} \mid \omega_n + \dot{\Phi}_0 \mid) \tag{45}$$

Now the complete theory must be analyzed. In the following paragraphs two different methods are used to obtain physical observables from this theory. These observables are very similar to those of the isolated island but the system they describe is fundamentally different and much more complicated. First use a method known as response theory is used. People have tried to use it for analyzing low the energy dynamics of the SET for decades. It will be possible to relate our calculations in the next paragraphs to this response theory approaches. The second method used is that of the physical observables which was used on the isolated island as well. An effective theory is formed by taking into account the response of the system to a perturbing background field ϕ. This gives insight into the methods that are used and makes it possible to interpreted the results.

5.2. Integer topological charge calculation.

First the system is studied with a fixed integer topological charge. For that reason $\phi = 0$ and $\Phi_0 = \omega_m \tau$. This is one of the instanton solutions of the action. It is possible to take this solution and use it to study the system. In the following steps an expansion in ω_m is made to obtain an effective action. The difference between the parameters of the old and new action will learn us the renormalization flow of the theory. First the definition is implemented in action (45). Then the variable of the theory is relabeled to m which results in the following action up to tree level:

$$S(m) = \frac{g}{2} \mid m \mid -2\pi iqm + \frac{\pi^2 m^2}{\beta E_C} \tag{46}$$

The structure of this effective action resembles parts that were found before for the isolated island. Now an effective action with modified parameters is sought. To lowest order in m one can also write:

$$S_{tot}(m) = S_{eff}(m) + K(i\omega_m) + K(i0) \tag{47}$$

Here S_{eff} is the same effective action as (13) with $\phi \to m$. Also the Matsubara correlator K is introduced, it is defined by the following integral:

$$K(i\omega_i) = \frac{-g}{4\beta} \int_0^\beta d\tau_1 d\tau_2 e^{i\omega_i \tau_{12}} \alpha(\tau_{12}) \langle e^{-i\Phi(\tau_1)+i\Phi(\tau_2)} \rangle \tag{48}$$

This is the first of many correlators that are used to find an analytical expression for the observable parameters of the theory. Correlators are used in this way because they only depend on the theory's structure by definition of the Renormalization Group. This means that they are not effected by renormalization. Since this correlator has nice analytic properties that the bare theory did not have. It can be used to find a result formulated in real frequency space that shows the total m dependence of the action. It is now possible to expand the theory in powers of ω_m and find the following effective action:

$$S_{tot}(\phi) = \frac{-g'}{2} \mid m \mid -2\pi imq' = \frac{\pi^2 m^2}{\beta E_C'} + \frac{2\pi i}{\beta F_C'} \mid m \mid m \tag{49}$$

Here the primed observable quantities are found by expressions containing the correlator above. Only the important $g^{'}$ and $q^{'}$ are presented since the other two are of higher order and much smaller.

$$g^{'} = 4\pi \mathrm{Im}(\frac{\partial K^R(\omega)}{\partial \omega} \tag{50}$$

$$q^{'} = Q + \mathrm{Re}(\frac{\partial K^R(\omega)}{\partial \omega} \tag{51}$$

Here $K^R(\omega)$ is the analytically continued Matsubara correlator: $K(i\omega_n)$ to $K(\omega + 0i) = K^R(\omega)$. These equations can the renormalized parameters of the theory if K^R is calculated. What is important to remember here is that they are found by perturbation theory on the action without a background field. This method of finding an expressions for the coupling constants in terms of correlators is known as general response theory.

5.3. Fractional topological charge calculation.

In this paragraph a different method will be used to find expressions for the renormalized coupling constants. This is now done with a recently developed method by Pruisken and Burmistrov [1, 18]. If one takes $\omega_n = 0$ then only the term proportional to ϕ is left in the kernel. This ϕ can only vary between plus and minus one half. Therefore it caries fractional topological charge. By taking $\omega_m = 0$ the investigation is restricted to the physics around $q = 0$ but the following analysis is also valid for other valued of k.

One would like to study the effect of small changes in ϕ. Therefore (45) is expanded in terms of ϕ. This gives a new action that can be expressed in a form that contains the normal two point correlator of the theory. First the two point correlator of the time dependent fields is written down and then the new action is presented.

$$D(i\omega_n) = T \int_0^\beta d\tau_1 d\tau_2 e^{i\omega_n \tau_{12}} \langle e^{-i\Phi(\tau_1) + i\Phi(\tau_2)} \rangle \tag{52}$$

$$S_{tot}(\phi) = S_{eff}(\phi) + \frac{g}{4\pi} \sum_n D(i\omega_n)(\mid \omega_n - 2\pi T\phi \mid - \mid \omega_n \mid) \tag{53}$$

Since this correlator depends on ω_n it is possible to isolate ϕ in the next step. This effective action yields again the same charge term as was found for the isolated island. In fact the action S_{eff} is the same as (13). It is possible to split the sum over n into two parts. One part where $n = 0$, which contains the renormalization of the conductance g. The second part for all other values of n, can be included in a renormalized definition of the external charge parameter q. This is then the effective action that is obtained:

$$S_{eff}(\phi) = \frac{g^{'}}{2} \mid \phi \mid -2i\pi q^{'}\phi + \mathcal{O}(\phi^2) \tag{54}$$

Here higher powers of ϕ are ignored because they are small and irrelevant to the further investigation. When the temperature goes to zero they will go to zero rapidly. The higher dimensional terms will be studied after we have explained the new primed variables. The two new variables are defined by the following equations:

$$\begin{aligned} g^{'} &= gTD(i\omega_0) \\ q^{'} &= Q - \frac{g}{2\pi}T\sum_{n>0}\mathrm{Im}D(i\omega_n) \\ Q &= q + \frac{i\langle\dot{\Phi}\rangle}{2E_C} = q - \frac{1}{2E_C}\frac{\partial\Omega}{\partial q} \end{aligned} \tag{55}$$

The renormalization of these two variables is enough to explain the SET systems parameter flow. For this approach to work the correlator $D(i\omega)$ must be calculated. The average charge of the island Q is now part of the new variable $q^{'}$. This new variable therefore is called the quasi-particle charge of the island. Recall that this was also the case of integer topological charge. It is a slight correction to the total charge that is found naturally when one defines the conductance as in (55). This new $g^{'}$ goes to zero along with the temperature, as do all the higher dimensional terms.

5.4. Higher dimensional terms.

Now the higher dimensional terms of the last paragraph will be presented to show some more detail. The terms of the integer section $\frac{1}{\beta E_C^{'}}$ and $\frac{1}{\beta F_C^{'}}$ are not given here but can be found in a recent paper by Burmistrov [19]. In the higher dimensional terms up to the third order a structure that contains the four point function is found. This is the four point correlator of action (45):

$$D(i\omega_n, i\omega_m) = T^2\int_{12}\int_{34} e^{i\omega_n\tau_{12}+i\omega_m\tau_{34}}\langle e^{-i\Phi(\tau_1+i\Phi(\tau_2))}e^{-i\Phi(\tau_3+i\Phi(\tau_4))}\rangle_{cum} \tag{56}$$

It contains the cumulant average of four events. This means that only connected diagrams are taken into account with this calculation. Below some of the first few higher order terms of (54) are given.

$$\mathcal{O}(\phi^2) = \frac{\pi^2\phi^2}{\beta E_C}\frac{\partial}{\partial q}(2q^{'} - Q) + \frac{g^2T^2}{8\pi^2}(D(0,0) - \sum_{n,m=0} Sgn(\omega_n\omega_m)D(i\omega_n, i\omega_m)) \\ + \frac{2\pi i\phi \mid \phi \mid}{4\beta E_C}\frac{\partial g^{'}}{\partial q} + \frac{ig^2T^2}{8\pi}\sum_{n\neq 0} Sgn(\omega_n)D(i\omega_n, 0) + \mathcal{O}(\phi^3) \tag{57}$$

They are all inversely proportional to β and therefore go to zero as the temperature does so. As before it is possible to define renormalized parameters that simplify the expressions. This is how the ϕ-dependence is made more explicit:

$$\mathcal{O}(\phi^2) = \frac{\pi^2}{\beta E_C^{'}}\phi^2 + \frac{2\pi i}{\beta F_C^{'}}\phi \mid \phi \mid + \mathcal{O}(\phi^3) \tag{58}$$

Now some final remarks on the different correlation functions will be made and their role within studying the SET is specified. Then they are calculated in two different regimes to find explicit formulations for the renormalized SET coupling constants.

5.5. Correlation functions.

The two point functions K and D have been used in two different ways to derive an effective action that contains physical observables. These observables are modified coupling constants in a new effective action. They are a function of correlators of the theory and they are therefore observable. In this section a more detailed study of these functions is made. First it is shown how one can write K in terms of D. Then the time dependent tunneling action is studied to find a good interpretation of what these correlators are. Below both correlation functions are shown for convenience:

$$D(i\omega_n) = T\int_0^\beta d\tau_1 d\tau_2 e^{i\omega_n\tau_{12}}\langle e^{-i\Phi(\tau_1)+i\Phi(\tau_2)}\rangle \tag{59}$$

$$K(i\omega_n) = \frac{-g}{4\beta}\int_0^\beta d\tau_1 d\tau_2 e^{i\omega_i\tau_{12}}\alpha(\tau_{12})\langle e^{-i\Phi(\tau_1)+i\Phi(\tau_2)}\rangle \tag{60}$$

Then the following two identities are used.

$$\int d\epsilon \frac{i\omega_m}{\pi(i\omega_m + \epsilon)} = \mid \omega_m \mid \tag{61}$$

$$D(i\omega_m) = \int d\epsilon \frac{\mathrm{Im}D^R(\epsilon)}{\pi(\epsilon - i\omega_m)} \tag{62}$$

The first is an identity from complex analysis and the second is the analytical continuation of the function D. When these identities are substituted into the second part of (60) the following happens:

$$\begin{aligned}
K(i\omega_n) &= \frac{-g}{4\pi\beta}\sum_{\omega_m}\int d\epsilon_1 d\epsilon_2 \frac{\mathrm{Im}D^R(\epsilon)}{\pi(\epsilon - i\omega_m)}\frac{\epsilon_2}{\pi(\omega_n+\epsilon_1+\epsilon_2)} \times [\frac{1}{\epsilon_1 - i\omega_m} + \frac{1}{\epsilon_2 + i\omega_1 + i\omega_2}] \\
K(i\omega_n) &= \frac{g}{4\pi^3\beta}\int d\epsilon_1 d\epsilon_2 \mathrm{Im}D^R(\epsilon_1)\frac{\epsilon_2}{(\omega_n+\epsilon_1+\epsilon_2)} \times \sum_{\omega_m}[\frac{1}{\epsilon_1 - i\omega_m} + \frac{1}{\epsilon_2 + i\omega_1 + i\omega_2}] \qquad (63) \\
K(i\omega_n) &= \frac{g}{4\pi^3\beta}\int d\epsilon_1 d\epsilon_2 \mathrm{Im}D^R(\epsilon_1)\frac{\epsilon_2}{(\omega_n+\epsilon_1+\epsilon_2)}[n_b(\epsilon_2) - n_b(\epsilon_1)]
\end{aligned}$$

In the last step the sum over ω_m resulted in two Bose-Einstein distributions functions; $n_b = \frac{1}{-1+e^{-\beta\epsilon}}$. This then needs to be analytically continued to get an expression based on real frequencies. Analytical continuation is always possible if a function is well defined in the complex plain. When the continuation is made a direction must be chosen in which the time evolves. Due to this choice only the retarded functions are chosen because only they make physical sense. Below the retarded function of K is given in terms of D^R

$$K^R(\omega) = g\int_{-\infty}^{\infty} d\epsilon_1 d\epsilon_2 \frac{\epsilon_2(n_b(\epsilon_2) - n_2(\epsilon_1))}{4\pi^2(\epsilon_1 - \epsilon_2 + \omega + i0^+)}\mathrm{Im}D^R(\epsilon_1) \qquad (64)$$

Here the 0^+ stands for the direction in which the continuation is taken and implies that the final result is the retarded function. Now the observables that were found by means of the first method can be expressed in terms of the correlator D as well:

$$q^{'} = Q + g\int_{\infty}^{\infty} d\epsilon \frac{\partial\epsilon n_b}{4\pi^2\partial\epsilon}\mathrm{Re}D^R(\epsilon) \qquad (65)$$

$$g^{'} = g\int_{\infty}^{\infty} d\epsilon \frac{\epsilon\partial n_b}{\pi\partial\epsilon}\mathrm{Im}D^R(\epsilon) \qquad (66)$$

Now the correlation function must be calculated to finish the study of the SET. Before this is done an analysis of the tunneling Hamiltonian is made to obtain insight in these equations. The SET can only be understood when one can relate the equations that are found back to the picture and original SET Hamiltonian. Below the tunneling part of this Hamiltonian is repeated:

$$\mathcal{H}_t^{(s)} = \sum_{k\alpha} t_{k\alpha}^2 a_k^{(s)\dagger} d_\alpha e^{-ieV^{(s)}t} + \sum_{k\alpha} t_{k\alpha}^{2\dagger} a_k^{(s)} d_\alpha^\dagger e^{ieV^{(s)}t} \qquad (67)$$

If one applies a voltage over the island $V^{(s)}$ will be nonzero: $V^l - V^r = V \neq 0$. One can define a current that flows from a reservoir into the island, or vise versa, in terms of the tunneling elements.

It looks as follows:

$$\mathcal{I}^{(s)} = e\frac{d}{d_t}\sum_k a_k^{(s)\dagger}a_k^{(s)} = -ie\mathcal{H}_t^{(s)} \tag{68}$$

If an expansion in the tunneling channels is made the current I can be rewritten in terms of correlators. this is valid up to the first order approximation. The second order term is of the order of $1/N_{ch}$. That is why one can say that the number of channels for now is taken to be infinite. Taking the continuum approximation and integrating the above expression up to a certain time t one finds:

$$\mathcal{I}^{(s)} = -i\int_{-\infty}^{t} dt' \langle[\mathcal{I}^{\alpha}(t), \mathcal{H}_t^{(s)}(t^t)]\rangle = -2e\text{Im}K^{(s)R}(ieV^{(s)}) \tag{69}$$

Here a little ambiguity comes up, understand that only the subscript t is to label tunneling and all others stand for time. Next it is necessary to specify the correlator K, it is given below in the first line. Below is its Matsubara frequency representation and in the lowest line it has been rewritten in terms of the normal correlator of the SET D.

$$\begin{aligned}
K^{(s)R}(\omega) &= i\int_0^{\infty} dte^{i\omega t}\langle[\sum_{k\alpha} t_{k\alpha}^2 a_k^{(s)\dagger} d_\alpha(t), \sum_{k\alpha} t_{k\alpha}^{2\dagger} a_k^{(s)} d_\alpha^\dagger(0)]\rangle \\
K^{(s)}(i\omega_n) &= \int_0^{\beta} d\tau e^{i\omega_n\tau} T\langle[\sum_{k\alpha} t_{k\alpha}^2 a_k^{(s)\dagger} d_\alpha(\tau) \sum_{k\alpha} t_{k\alpha}^{2\dagger} a_k^{(s)} d_\alpha^\dagger(0)]\rangle \\
K^{(s)}(i\omega_n) &= \frac{-g^{(s)}}{4\beta}\int_0^{\beta} d\tau_1 d\tau_2 e^{i\omega_n\tau_{12}}\alpha(\tau_{12})D(\tau_{21})
\end{aligned} \tag{70}$$

In the last equation one can clearly see how from $K^{(R)}$ the kernel α is extracted and the D is isolated. If one compares the last equation with the original correlator K that was introduced in the integer topological charge section the following ratio is obtained:

$$K^{(s)}(i\omega_n) = \frac{g_s}{g}K^R(i\omega_n) \tag{71}$$

This holds for real arguments as well. The real part of the retarded function is naturally given by the following formula:

$$\text{Re}K^{(s)R}(-eV^{(s)}) = \frac{i}{2e^2}\int_{-\infty}^{t} dt' \langle[\mathcal{I}^{(s)}(t), \mathcal{I}^{(s)}(t')]\rangle \tag{72}$$

On the right hand side the current-current-correlator pops up unexpectedly. One can write it down in a more general way as $S_I(\omega, V) = \int_{-\infty}^{\infty} dte^{-i\omega t}\langle[I(t), I(0)]\rangle$. Then it is used to find a new way to express q' in terms of the correlators:

$$q' = Q + \frac{(g_l+g_r)^2}{g_l g_r} \times \text{PrincipalValue}[\int d\omega \frac{1}{2\pi\omega}\frac{\partial S_I(\omega, V)}{\partial V}] \tag{73}$$

One can use the retarded function as well to derive the conductance of the SET. The current is expressed in terms of K so it is possible to use the relation $I = G \times V$. This allows one to find the SET's total conductance G.

$$G = \frac{g_l g_r}{(g_l + g_r)^2} g^{'} = \frac{g_l g_r}{(g_l + g_r)^2} g \int_{-\infty}^{\infty} d\epsilon \frac{\epsilon \partial n_b}{\pi \partial \epsilon} \mathrm{Im} D^R(\epsilon) \tag{74}$$

The above expression is given in natural units of $e^2/\hbar$. This means that the observable $g^{'}$ completely determines the conductance through the island. This is why $g^{'}$ and $q^{'}$ will be used to find the RG flow for all values of these variables. In this paragraph K^R was related to D^R. Also the physical observables in our formula's were related to macroscopic entities of the system.

6. WEAK COUPLING ANALYSIS.

In the next two sections the correlator $D(i\omega)$ will be calculated. In the first section this will be done for the part of the theory where g is large. In the next section the correlator is calculated for low values of the conductance parameter. It is not possible to find an analytic formulation that contains all the relevant renormalization behavior in both regimes. One will be able to match the resulting equations of both sides in the last paragraph of the theory section.

6.1. Perturbation Theory in $1/g$.

The regime of large values of g is typically called the weak coupling limit. This name comes from high energy physics where similar theory's are discussed with a corresponding parameter $1/g$. Going back to the SET's picture one can think of this limit as the regime where many electrons tunnel through the island. From the outset of the problem it is therefore unlikely that any of the quantities of this problem are quantized. Since g is large one can expand the action of the AES-theory in powers of $1/g$.

$$\begin{aligned}
S_0 &= (g/4)\int_0^\beta d\tau_1 d\tau_2 \frac{T}{\pi}\sum_{\omega_n} \mid \omega_n \mid (1+i\phi_n - i\phi_n^2)(1-i\phi_{-n}-i\phi_{-n}^2) + \int_0^\beta d\tau (2\pi T n)^2/E_C \\
&= (g/4)\int_0^\beta d\tau_1 d\tau_2 \frac{T}{\pi}\sum_{\omega_n} \mid \omega_n \mid (i\phi_n)(-i\phi_{-n}) + \int_0^\beta d\tau (2\pi T n)^2/E_C \sum_n \phi_n\phi_{-n} \\
&= (g/4)\sum_{n\rangle 0} \frac{T}{\pi}(\pi n T\beta^2)(i\phi_n)(-i\phi_{-n}) + \int_0^\beta d\tau (2\pi T n)^2/E_C \sum_n \phi_n\phi_{-n} \\
&= \sum_{n\rangle 0} [(gn) + (2\pi T n)^2/E_C](i\phi_n)(-i\phi_{-n})
\end{aligned} \tag{75}$$

Here perturbation theory is used and $\sum_n \phi_n\phi_n = 1$ was added to the second term. The last line can be simply notated as:

$$S_0 = g\sum_{n\rangle 0}(n + \frac{2\pi^2 T n^2}{gE_C})\Phi_n\Phi_{-n} \tag{76}$$

One recognizes this formulation from the number of particles representation (8). This is all one needs to calculate the correlator of this theory. In general the correlator is $D = \langle\psi(\tau_1)\psi(\tau_2)\rangle$. Here the average is calculated by the functional integral $\langle ...\rangle = \frac{\int D\phi e^{-S}(...)}{\int D\phi e^{-S}}$. Performing this calculation with the action above gives:

$$D(i\omega) = \beta(1 - \frac{2}{g}\sum_{s\rangle 0}\frac{1}{s + 2\pi^2 T s^2/gE_C})\delta_{n,0} \\ + 2\pi i(1-\delta_{n,0})(\frac{1}{ig\mid\omega_n\mid} - \frac{1}{ig\mid\omega_n\mid + ig^2E_C/\pi}) \tag{77}$$

Using $\delta_{n,0} = \lim_{\eta\to 0}[\eta(i\omega_n+\eta)^{-1}]$ one can analytically continue this result to real frequencies ω. As before one is only interested in the retarded correlator since it places τ_2 later then τ_1 and therefore makes physically sense.

$$D^R(\omega) = \beta(1 - \frac{2}{g}\ln[\frac{gE_Ce^\gamma}{2\pi^2 T}])\lim_{\eta\to 0}\frac{\eta}{\omega+\eta+i0^+} \\ + \frac{2\pi i}{g}(\frac{1}{\omega+i0^+} - \frac{1}{\omega+igE_C/\pi}) \tag{78}$$

This allows one to solve for the first time the equations for the physical observables. To shorten this study this is postponed for a while. First the non-perturbative effects of instantons are taken into account. When the instanton correlator has been found it is possible to calculate all the physical observables in the weak coupling regime at once.

6.2. Instantons at large g.

Recall that the AES action contains stable minima at large values of the coupling constant g. This is why one needs to take these instantons into account. The integrals found in (23) can now be taken into account because they are now studied in the weak coupling regime. First recall Ω_{inst}:

$$\beta\Omega_{inst} = -D\int_0^\beta d\tau_0\int_0^\beta\frac{d\lambda}{\lambda^2}g(\lambda)e^{-g(\lambda)/2+(2T-4/\lambda)/E_C(\lambda)}\cos(2\pi q) \tag{79}$$

Solving this has been done before many times [20–22]. This gives the following expression for the instanton thermodynamic potential:

$$\beta\Omega_{inst} = -\frac{g^2}{\pi^2}\beta E_C e^{-g/2}\ln[\frac{\beta E_C}{2\pi^2 e^\gamma}]\cos(2\pi q) \tag{80}$$

The expression for the average charge Q in terms of Ω (55) can be used to calculate this. After straight forward differentiation one finds:

$$Q(T) = q - \frac{g^2}{\pi}e^{-g/2}\ln[\frac{E_C}{2\pi^2 e^\gamma T}]\sin(2\pi q) \tag{81}$$

The corrections to $Q = g$ go with g^2e^{-g} and are periodic in q with a period $2\pi q$. Although this already gives some hints of the parameter $q^{'}$ one still needs to calculate the instanton correlator to find explicit expressions for $q^{'}$ and $g^{'}$. To find the correlator the operator specific parts of (23) needs to be calculated. Since the solutions for $W=0$ are already taken into account in the last paragraph one only needs to select the parts that depend on non-trivial winding number, $W \geq 1$. Recall that $\lambda = \beta(1- \mid z_1 \mid^2)$ is the scale size of an instanton event with $W=1$. This is all one needs to extract relevant parts of Eq. (23):

$$\int_0^\beta d\tau_0(\mathcal{O}(\Phi) - \langle \mathcal{O} \rangle_0) = \beta[-\frac{\lambda}{\beta}\delta_{n,0} + \frac{\Theta(nW)\lambda^2}{\beta^2}(1-\frac{\lambda}{\beta})^{(|n|-1)}] \tag{82}$$

Here $\mathcal{O}$ is any winding number depended operator, for instance the correlator $D(i\omega_n)$. One recognizes the n dependence in all parts and Θ is the Heaviside step function. If one then integrates over the scale size λ the instanton correlator is found:

$$D_{inst}(i\omega_n) = \frac{-g^2 E_C}{\pi^2 T^2} e^{-g/2}[\delta_{n,0}\cos(2\pi q) - i\pi T e^{2\pi i q Sgn(n)}(1-\delta_{n,0})(\frac{1}{i \mid \omega_n \mid} - \frac{1}{i\omega_n + 2\pi i T})] \tag{83}$$

Analytically continuing this to real frequencies gives:

$$D^R_{inst}(\omega) = \frac{-g^2 E_C}{\pi^2 T^2} e^{-g/2}(\cos(2\pi q) \lim_{\eta \to 0} \frac{\eta}{\omega + \eta + i0^+} - i\pi T e^{2\pi i q}(\frac{1}{\omega + i0^+} - \frac{1}{\omega + 2\pi i T})) \tag{84}$$

Now two correlators are found that are relevant to the weak coupling limit. These will be used to calculate the physical observables that were found before in the paragraph on integer topological charge. The integer approach results can be related to the response theory results that were obtained before in this regime. The simpler approach via the field ϕ yields the same results up to a small numerical factor that will be discussed at the end of the section. The results that were found by studying the system with integer topological charge are repeated below:

$$q^{'} = Q + g\int_{\infty}^{\infty} d\epsilon \frac{\partial \epsilon n_b}{4\pi^2 \partial \epsilon} \mathrm{Re} D^R(\epsilon) \tag{85}$$

$$g^{'} = g\int_{\infty}^{\infty} d\epsilon \frac{\epsilon \partial n_b}{\pi \partial \epsilon} \mathrm{Im} D^R(\epsilon) \tag{86}$$

For the instantons this gives:

$$g^{'}_{inst} = \frac{-g^3 E_C e^{-g/2}}{6T}\cos(2\pi q) \tag{87}$$

$$q^{'}_{inst} = Q(T) - \frac{g^3 E_C e^{-g/2}}{24\pi T}\sin(2\pi q) \tag{88}$$

and after substituting in the correlator that was found by perturbing the AES-theory in the weak coupling regime, (78) the observables are:

$$g^{'}(T) = g - 2\ln[\frac{g E_C e^{\gamma+1}}{2\pi^2 T}], \qquad q^{'}(T) = q \tag{89}$$

Here the integral $\int_0^\infty \frac{dx}{x^2+\pi^2 z^2} \frac{x^2}{\sinh^2 x} = \frac{1}{2|z|} - 1+ \mid z \mid \psi^{'}(1+ \mid z \mid)$ is used. Adding the two contributions up one finds the following weak coupling results:

$$g^{'}(T) = g - 2\ln\frac{gE_C e^{\gamma+1}}{2\pi^2 T} - \frac{g^3 E_C e^{-g/2}}{6T}\cos(2\pi q) \tag{90}$$

$$q^{'}(T) = q - \frac{g^3 E_C e^{-g/2}}{24\pi T}\sin(2\pi q)(1 + \frac{24T}{gE_C}\ln\frac{E_C}{2\pi^2 e^{\gamma} T}) \tag{91}$$

When these results are studied one notices a few remarkable things. First of all one sees that the corrections to both original parameters are still periodic in q. Secondly that these variations are much larger then the correction to the average charge Q. The logarithm gives a new restrained on the validity of these equations, $T \gg g^3 E_C e^{-g/2}$. All corrections are proportional to the inverse temperature making this an important relation. The relation between g and T will be studied more explicitly later after the strong coupling observables have been calculated. One must remember that these results are obtained by calculations up to first order in $1/g$. This section will end with a small paragraph on the renormalization flow of these physical observables.

6.3. Renormalization flow.

It is possible to extract the temperature dependence of the original conductance parameter $g(T)$ to simplify the expressions we found:

$$g^{'}(T) = g(T) - Dg^2(T)e^{-g(T)/2}\cos(2\pi q) \tag{92}$$

$$q^{'}(T) = q - \frac{Dg^2(T)}{4\pi}e^{-g(T)/2}\sin(2\pi q) \tag{93}$$

Here a new numerical constant $D = (\pi^2/3)e^{-\gamma-1} \approx 0.68$ was introduced that absorbed some of the constants in the equations. Also $g(T)$ is introduced:

$$g(T) = g - 2\ln\frac{gE_C}{6DT} \tag{94}$$

This gives the temperature dependence of the original conductance parameter g. Notice the resemblance with the perturbation result a few lines above (89). These two equation must contain the same physics since the perturbative renormalization of the AES action must give the temperature dependence of the AES coupling constants.

Furthermore, the equations (92), (93) can also be found by the mathematically simpler equations of the fractional field approach (55). The fact that both approaches present the same physics ensures that they are well understood and that no mistakes have been made. If one would do all the mathematics one would find a different numerical factor $D = 2e^{-\gamma} \approx 1.097$ and in (94) $E_C \rightarrow (\frac{6}{\pi^2})E_C$. These small numerical difference do not change the physics. Constant factors do not influence the renormalization flow and will always appear when an effective theory is constructed to study renormalization. Note that this is only valid in the weak coupling regime where no critical points are found. The same approaches will be studied at the strong coupling side to see if they produce the same results there as well.

Finally the RG equations are presented in differential form below together with a figure where the corresponding parameter flow is given.

$$\beta_g = \frac{dg'}{d\ln\beta} = -2 - 4/g' - g'^2 D e^{-g'/2} \cos(2\pi q') \tag{95}$$

$$\beta_q = \frac{dq'}{d\ln\beta} = -g'^2 \frac{D}{4\pi} e^{-g'/2} \sin(2\pi q') \tag{96}$$

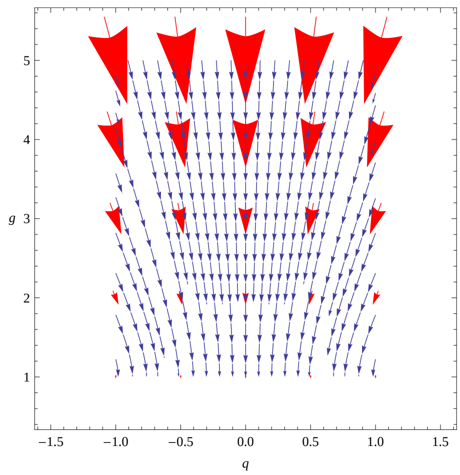

FIG. 17: This is a graph of the RG flow around $q = 0$ for large g.

Here the conductance g is plotted on the vertical axes. In the picture the RG flow is plotted around $q = 0$. The flow is periodic in q so one finds this flow for all values of q going down. Now the parameter flow at low values of the conductance parameter g will be studied. Then both regimes will be combined to make a single plot.

7. STRONG COUPLING ANALYSIS.

Now the other side of the g-axes is considered to calculate the physics of the SET with very weak tunneling contacts. The goal of this part of the project is to derive the exact renormalization behavior of the SET near $g = 0$ at integer values of k or $q = 0.5$. this study will limit the calculations to the physics around $\theta \approx \pi$. Therefore one is interested in the small g limit which is counter intuitively known as the strong coupling limit from High Energy Physics (HEP). An useful analogy to the Kondo problem, discovered by Matveev[11] will be introduced. This Kondo Hamiltonian is solved in the limit that is of interest to the strong coupling SET. The Kondo problem is the description of a magnetic impurity in an conduction environment of electrons with spin. To describe a system like that the grand partition function formalism is used as before. The Hamiltonian of the theory near $g = 0$ and $\theta = \pi$ is given below:

$$\begin{aligned} H &= H_0 + H_c + H_t^l + H_t^r \\ H_0 &= (\sum_k \epsilon_{k,L} a_{k,L}^\dagger a_{k,L}) + (\sum_k \epsilon_{k,R} a_{k,R}^\dagger a_{k,R}) + (\sum_\alpha \epsilon_\alpha d_\alpha^\dagger d_\alpha) \\ H_c &= E_c(k-q)^2 + \frac{\Delta}{2} - \Delta S_z \\ H_t^s &= \sum_{k,\alpha} t_{k,\alpha}^s a_k^{s\dagger} d_\alpha S^+ + \sum_{k,\alpha} t_{k\alpha}^{s*} a_k^s d_\alpha^\dagger S^- \end{aligned} \tag{97}$$

Here $S^\pm = S^x \pm iS^y$ and $\Delta = E_c(1 - \frac{\theta}{\pi})$ is the energy gap of the SET.

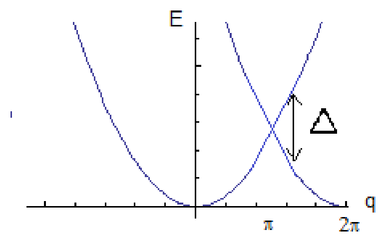

FIG. 18: Here the Coulomb potential and Δ as a function of θ are plotted

This Hamiltonian can be transformed in a partition function by using a formalism first introduced by Abrikosov. It is nowadays known by the Abrikosov pseudo-fermion technique [23]. One introduces two component field ψ and corresponding $\bar{\psi}$ to describe the electrons. Using the symmetry of the problem with these pseudo-fermion fields one arrives at the following action:

$$S = \beta E_C(k-q)^2 + \frac{\beta\Delta}{2} + \int_0^\beta d\tau \bar{\psi}(\partial_\tau - \eta + \frac{\Delta\sigma_z}{2})\psi + \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{12})[\bar{\psi}(\tau_1)\sigma_-\psi(\tau_1)][\bar{\psi}(\tau_2)\sigma_+\psi(\tau_2)] \tag{98}$$

Here the Pauli matrices $\sigma_i, i = x, y, z$ and $2\sigma_\pm = \sigma_x \pm i\sigma_y$ are used. A chemical potential is introduced in the grand partition function. It must be taken out in the end of all calculations to ensure that there are only the tunneling events of a single electron. The fermion fields allow for four different particle-like configurations of the fields. This is shown in Figure 19:

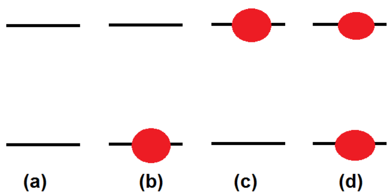

FIG. 19: Here the possible field configurations described by the action (98) are given.

In the first configuration nothing happens so it must be eliminated. One only wants to describe a single tunneling particle so the last scenario should be eliminated as well since it has more than one pseudo-fermion. If one performs the following limit in the end of all calculations only the physical events are kept:

$$Z = \lim_{\eta \to -\infty} \frac{\partial Z_{pf}}{\partial e^{\beta\eta}} \tag{99}$$

This limit ensures that only events with $N_{pf} = 1$ are taken into account by multiplying other events with 0. Not only need a good partition function is needed but also good operators and observables are required to make physical sense. Therefore one must also perform these steps when calculating expectation values and this is done as follows:

$$\langle \mathcal{O} \rangle = \lim_{\eta \to -\infty} \left(\frac{Z_{pf}}{Z} \frac{\partial \langle \mathcal{O} \rangle_{pf}}{\partial e^{\beta\eta}} + \langle \mathcal{O} \rangle_{pf} \right) \tag{100}$$

The averages within the limit are taken with respect to the pseudo-fermion theory. One can use this theory to calculate the renormalization flow at low values of g. The theory of (98) is the same as that of the XY-Kondo model. This analogy will can help one to understand the calculations and allows for the comparison of the two different systems. This analogy has been known for some years and the relation was first described by Matveev in 1991 [11]. In the Kondo model the magnetization of an electron with spin is calculated in a background of other electrons. Due to the spin the Kondo model has a Zeeman term that one can relate to the energy gap of our AES-theory. They have the same twofold symmetry and describe energy differences between two energy levels of a system.

From the interaction part of the action one can conclude that the spin operators $[\bar{\psi}(\tau)\sigma_{\pm}\psi(\tau)]$ are a different representation of the AES instanton operators $e^{\pm i\Phi}$ acting on the $Q = k$ and $Q = k + 1$ states of the isolated island. They represent events in which q increases by an integer. It is important to remember that this action is only valid in the regimes where $g \ll 1$, $\mid q - k - 0.5 \mid \ll 1$ and $\beta E_C \gg 1$.

7.1. Derivation of $D(i\omega)_{strong}$.

To obtain the physical observables of the SET near $g = 0$ one must find the correlator of the theory in this regime. This is done in the one loop renormalization procedure also known as the leading logarithmic approach.

7.1.1. The self-energy.

To get some insight in the new action it is briefly studied without conductance. If $g = 0$ the Greens function of the theory is easily obtained from the action.

$$G_{0\pm}^{-1}(n) = i\pi T(2n + 1) + \eta \mp \frac{\Delta}{2} \tag{101}$$

One can express this Greens function as well in terms of the self-energy. This corresponds to the most simple diagram of an interaction with it self. This is the diagram for the theory's self-energy:

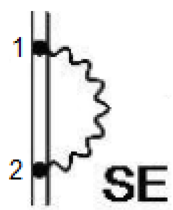

FIG. 20: A diagram of the self energy where the solid line is the interaction line.

In this figure time moves from north to south. The expression for the Greens function then becomes:

$$G_{\pm}^{-1}(n) = i\pi T(2n + 1) + \eta \mp \frac{\Delta}{2} - \Sigma_{\pm}(i\pi T(2n + 1)) \tag{102}$$

Here $\epsilon_n = \pi T(2n+1)$ is the energy levels variable. The renormalization of the self-energy can be calculated in terms of two vertex function belonging to the two points of self-interaction. The result of this renormalization calculation will then depend on the high energy cut-off that is chosen when calculating the two vertex functions:

$$\gamma(x) = \gamma_s^{-1}(x) = (1 + \frac{gx}{2\pi^2})^{1/2} \tag{103}$$

with $x = \ln[\Delta\gamma/\gamma_s, |i\epsilon_n + \eta|]$. Here γ corresponds to the first interaction moment(1) and its inverse γ^{-1} to the second in time(2). The pseudo-fermion Greens function in terms of these vertex functions are then:

$$G_{\pm}^{-1}(i\epsilon_n) = (i\epsilon_n + \eta)\gamma(i\epsilon_n) \mp \gamma_s(i\epsilon_n)\frac{\Delta}{2} \tag{104}$$

7.1.2. Partition function and $\langle Q \rangle$.

In this paragraph the self energy calculation will be used to rewrite the partition function in terms of the Greens function of the theory. For now g is assumed to be zero and by doing these calculations one will see what this assumption gives. If one uses (98) the partition function that describes only the physics of the SET without interactions is given by:

$$Z = e^{\beta(k-q)^2 + \beta\Delta/2} \lim_{\eta \to -\infty} e^{-\beta\eta} \sum_{n,\sigma=\pm} e^{i\pi T(2n+1)0^+} G_\sigma(n) \tag{105}$$

Here $\Delta = (1 - \theta/\pi)E_C$ is the energy gap of the SET as before. One can perform the summation since the vertex functions in G_σ^{-1} are known. This results in the simple expression below.

$$Z = 2e^{\beta(k-q)^2 + \beta\Delta/2} \cosh\frac{\beta\Delta}{2\gamma'} \tag{106}$$

Here a new renormalization factor γ is introduced:

$$\gamma' = (1 + \frac{g}{2\pi^2}\ln(\frac{\Lambda}{max(\Delta/\gamma^2, T)}))^{1/2} \tag{107}$$

The renormalization factor is called like this because it changes the values, in this case the energy gap Δ, of the original theory. One can use the partition function to derive the average charge of the SET $\langle Q \rangle$ or in terms of the Kondo model the average magnetization M.

The two can be related by $Q(T) = 0.5 + k - M(T)$. From Kondo model studies [11] it is known that $M(T) = \langle S_z \rangle$. Simply calculating the quantum mechanical average results in the following magnetization:

$$M(T) = \frac{1}{2\gamma^2}\tanh(\frac{\beta\Delta}{2\gamma^2}) \tag{108}$$

This expression was originally found in one of the older studies to the Kondo model in Ref. [24, 25]. When one looks at the limit in which $T = 0$ the following formula is found:

$$Q(T=0) = k + \frac{1}{2} - \frac{1}{1 + \frac{g}{2\pi^2 \ln(\Lambda/\Delta')}} \tag{109}$$

One can see that this is therefore no longer quantized for finite values of g. Now finite values of g in our Hamiltonian will be considered and the corresponding correlator of that theory will be derived to study this in more detail.

7.1.3. The correlation function $D^R(\omega)$.

The derivation of the self energy was not enough to obtain all the required physics at the strong coupling side of the problem. Just as in the case of weak coupling one needs the correlator on the strong coupling side to derive the physical observables. The calculation of the average charge by means of the self energy has given some insight in the physics that can be suspected and showed in a simple way the analogy with the Kondo model. The two-point correlator can be drawn in a diagrammatic way as is sketched in Figure 21. This is the lowest order contribution to the vertex function due to the complex coupling of the tunneling interaction.

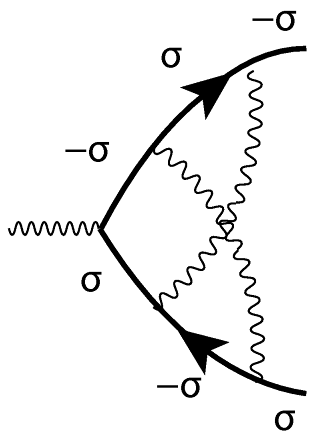

FIG. 21: This is the first contribution to the full vertex function.

As before one needs to go from pseudo-fermions to the real SET electrons which accounts for the summation over two Greens functions in the equation below.

$$
\begin{aligned}
D(i\omega_n) &= -\frac{1T}{2\cosh\beta\Delta'/2}\lim_{\eta\to-\infty}\frac{\partial}{\partial e^{\beta\eta}}\sum_{\epsilon_m}G_-(i\epsilon_m)G_+(i\epsilon_m+i\omega_n) \\
&= -\frac{1}{(i\omega_n-\Delta')\gamma^2}\tanh(\beta\Delta^{'}/2)
\end{aligned}
\tag{110}
$$

In the second line the summation has been performed as before. After analytic continuation one obtains:

$$
D^R(\omega) = -\frac{1}{(\omega-\Delta'+i0^+)\gamma^2}\tanh(\beta\Delta^{'}/2) \tag{111}
$$

This is then the final correlator that can be used to study the physics at the strong coupling side. It is inversely proportional to the renormalization factor as one might aspect. Also it has periodic divergences for certain values of ω and $\Delta^{'}$ hinting on periodic phase transitions. In the following paragraphs these will be studied in full detail.

7.2. Physical observables at strong coupling.

Now that the correlator on the strong coupling side has been found one can calculate the renormalized coupling constants. Recall that two different methods have been used to calculate these parameters. Both definitions give the same result for the quasi-charge of the SET $q^{'}$:

$$
q^{'} = Q + \frac{1-\gamma^2}{2\gamma^2}\tanh(\beta\Delta^{'}/2) = k(q) + \frac{1}{e^{\beta\Delta^{'}}+1} \tag{112}
$$

As expected the quasi-charge resembles the original definition very much. When the temperature is equal to zero the exponent is equal to 1 and $q^{'}(T=0) = k(q) + 1/2 = q$. In the following plot the quantized charge Q is plotted at zero(black) and finite(red) temperature:
The results for the quasi-conductance $g^{'}$ are more interesting. The two different methods that were used do not give the same results. For the method of physical observables one finds:

$$
g_I^{'} = \frac{g}{2\beta\gamma^2\Delta'/2}\tanh(\beta\Delta^{'}/2) \tag{113}
$$

Here $\Delta^{'}$ is the renormalized gap that was introduced before. The expression above shows a periodic divergence when the gap closes. This is a feature one also encounters when the correlator is filled in the equation that was found by the first response theory method:

$$
g_{II}^{'} = \frac{g\beta\Delta^{'}}{2\gamma^2}\frac{1}{\sinh(\beta\Delta'/2)} \tag{114}
$$

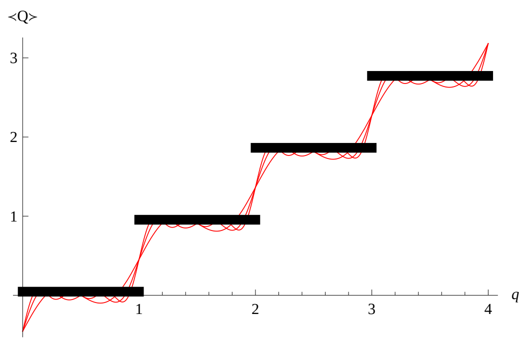

FIG. 22: This is a function of the macroscopically quantized charge on the island at zero(black) and finite(red) temperature.

This result is found by methods that were already known. It appeared in another but similar context in Ref. [26]. Several observations can be made when looking at these equations. First of all one sees that they completely match the equations of the isolated island when $T = 0$. This is a remarkable feature since both systems are macroscopically very different. Some parts of the SET's Hamiltonian are also present in the isolated island but there is no tunneling in that case. The isolated island was more a though experiment then a real system. The SET at zero temperature is a real system and this makes it totally different. In the zero temperature limit of the SET there are still two nearby reservoirs from which electrons can tunnel. Already is was shown that $q^{'}$ is robustly quantized, a feature of the SET that was not known before the recent publications by Pruisken and Burmistrov [1].

Although the different methods give different equations for $g^{'}$ they are in fact equal for small values of $\beta\Delta^{'}$. This therefore determines the validity of the one loop renormalization that was used to find these results. One can also state that this strong coupling expansion does not give any knowledge about the Coulomb blockade phases of the SET. For these phases around $\theta = \pi$ the theory has a large energy gap and $\beta\Delta^{'} \gg 1$. The regime where $\beta\Delta^{'} \ll 1$ is luckily more interesting because it contains a critical point that will be studied in the next paragraphs.

7.3. Renormalization at zero temperature.

It was shown that two renormalization procedures are necessary to describe the SET. These renormalization procedures make it possible to study the low energy dynamics of the SET or the corresponding high energy structure of the Kondo model. At zero temperature the following renormalization equations are obtained:

$$g^{'} = \frac{g}{\gamma^2} = \frac{g}{1 + \frac{g}{2\pi^2}\ln\frac{\Lambda}{\Delta'}} \qquad \Delta^{'} = \frac{\Delta}{\gamma^2} = \frac{\Delta}{1 + \frac{g}{2\pi^2}\ln\frac{\Lambda}{\Delta'}} \tag{115}$$

This corresponds to the following equations in differential form

$$\beta_g = \frac{dg}{d\ln\Lambda} = g^2/2\pi^2 \qquad \gamma_\Delta = \frac{d\ln\Delta}{d\ln\Lambda} = g/2\pi^2 \tag{116}$$

From these results it follows that there is no renormalization for $g = 0$. This means that the case with no conductance is fundamentally different then the SET with tunneling. This emphasizes again the fact that Q is no longer quantized in the conducting case. Or in Kondo terms that there is no spontaneous magnetization for finite values of the spin interaction. One can conclude that the SET at $T = g = 0$ is a limited theory with special and simple renormalization. This gets much more complicated when the results at finite temperatures are taken into account.

7.4. Strong coupling results at finite temperature.

In the case where $\beta\Delta^{'} \ll 1$ the results that were obtained from the theory of physical observables agree with those found by response theory. Now the renormalization in the strong coupling limit is analyzed. After that a complete phase diagram for the SET will be obtained. As before the renormalization equations for the coupling constants can be written down:

$$g^{'}(T) \simeq g(1 + g\ln[\beta\Lambda]/2\pi^2)^{-1}/2 \tag{117}$$

$$q^{'}(T) \simeq k + 0.5 - \beta\Delta(1 + g\ln[\beta\Lambda]/2\pi^2)^{-1}/4 \tag{118}$$

Here Λ is the energy cut off that was introduced in the strong coupling vertex function. These functions can be put into a differential form.This is done along $q^{'} = k+0.5$ to stay in the appropriate regime. By this means the following results are found:

$$\beta_g = \frac{dg^{'}}{d\ln\beta} = -g^{'2}/\pi^2 \tag{119}$$

$$\beta_q = \frac{dq^{'}}{d\ln\beta} = (q^{'} - k - \frac{1}{2})(1 - \frac{g^{'}}{\pi^2}) \tag{120}$$

Here a remarkable feature attracts the attention. These equations contain the same critical fixed point that was found for the isolated island (30). Although the systems are macroscopically very different they have a similar behavior at low energies and conductances. The conductance parameter in this regime has become marginally irrelevant as one can see from the minus sign in the equations.

Since only the physics near $\theta = \pi$ can be considered the Coulomb blockade can not be made explicit here. However the strong similarity of these results with those of the isolated island and the structure along $g^{'}$ allow one to make an educated guess. Given the fact that the theory's energy gap becomes very large when $\theta \approx 0$ one expects a similar flow as near the other critical point of (30). The equations cannot be obtained by the methods used here but will most probable have the following form:

$$\beta_g = g^{'}\ln g^{'} \qquad\qquad \beta_q = (q^{'} - k)\ln \mid q^{'} - k \mid \tag{121}$$

With these β-functions combined with the weak coupling results it is possible to find a complete phase diagram of the SET. This will be done in the next section.

8. THE COMPLETE RENORMALIZATION DIAGRAM.

Now that the renormalization flow at high and low conductances has been found it is possible to combine the two in a single plot. In this section an unifying scaling diagram for the SET in terms of the two quasi-coupling constants $g^{'}$ and $q^{'}$ is constructed and analyzed. First the differential equations for these parameters are given in a single plot. In Figure 23 β_g is plotted for low and high values of the conductance:

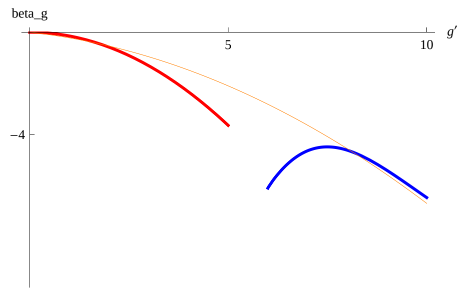

FIG. 23: Here the weak and strong coupling β-equation is plotted for g.

An interpolation between the two graphs can be made in orange to match the larger part of the functions in both regimes. For the quasi-charge a similar plot can be made:

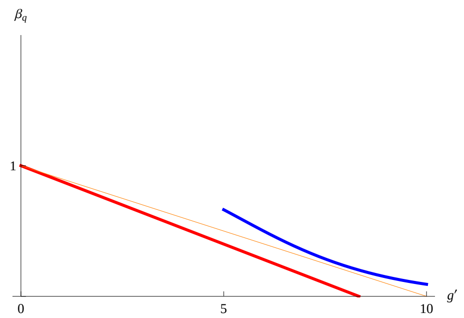

FIG. 24: Here the weak and strong coupling β-equation is plotted for q.

The two functions seem to match rather nicely with the intermediate orange line in their corresponding regimes. This then allows one to combine the flow diagrams that were found for the weak coupling and instantons calculation with the flow near $g = 0$. Here the final plot of this investigation is drawn.

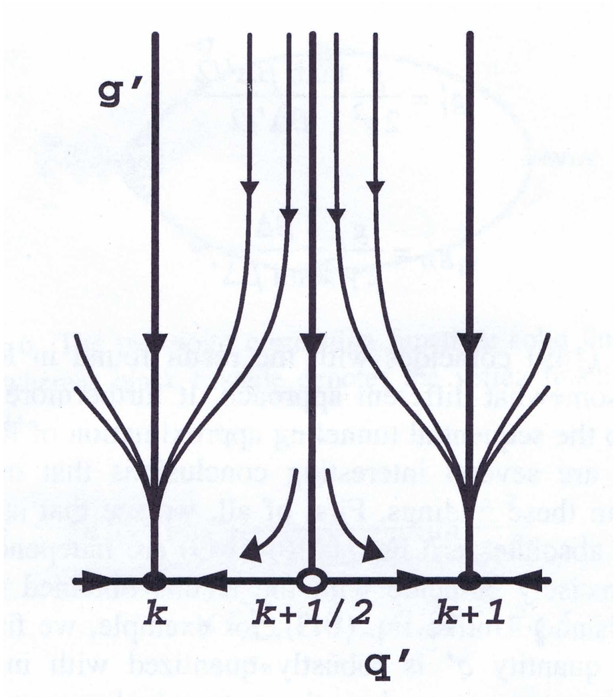

FIG. 25: This diagram gives the unified scaling behavior of the SET. The arrows show scaling for decreasing temperature.

As has been said many times the strong coupling analysis was only made in the regime near half integer values of q but due to the direction of flow there is little doubt about the validity of the flow near integer k. This plot clearly shows the critical fixed points at $g^{'} = 0$ and $q^{'} = k + 0.5$. Also the stable fixed points at integer values of $q^{'}$ can be seen along the line of $g = 0$. Therefore the flow of the SET at zero conductance resembles the isolated island very well. The SET has a robustly quasi-charge which is remarkable for a system that can be seen by the naked eye. This is a feature that is not clear when one studies the SET when the conductance is approaching zero. It is possible to relate the variable $g^{'}$ to the SET conductance and the quasi-charge to the antisymmetric current-current-correlator. These relations are a viable entry point to study the SET experimentally as is currently done at the Aalto University in Helsinki.

This is where the current section of this project ends. The remainder of this report is dedicated to redoing some of the calculations when taking into account an extra order in perturbation theory.

9. NEXT ORDER CALCULATION OF THE TUNNELING ACTION.

The goal of this section is to find the kernel that corresponds to an extra order correction to the SET's action. To derive this first the details of the second order kernel calculations are presented. This is a necessary step to then calculate the fourth order correction. When the details of the second and fourth order calculation are found, the differences between the two parts of the action can be studied. Continuing, the newly found correction will be used to study its effect on the physics that was derived in this paper. First a study is made of the effects on the instanton solutions that existed for the action up to second order. When these calculations are done the conclusions of this research project are made.

9.1. The kernel α.

Recall the SET's action in terms of tunneling channels, Eq. Now the same action can be represented in a matrix formulation in the following manner:

$$S_{tun} = TrLog \begin{bmatrix} G_{\alpha}^{-1} & te^{i\phi} \\ e^{-i\phi}t^{\dagger} & G_{a}^{-1} \end{bmatrix} \tag{122}$$

Here $G_{i,n}^{-1} = i\omega_n + \mu - \epsilon_i$ and $\omega_n = 2\pi(n)T$. The matrices have a dot-lead structure and that is why the tunneling elements are off-diagonal. The dot and lead quantum numbers are α and a, or $i = \{\alpha, a\}$. Since the tunneling elements are generally small one can make an expansion in them. First the off-diagonal matrix is extracted:

$$S_{tun} = TrLog(\begin{bmatrix} G_{\alpha,n}^{-1} & 0 \\ 0 & G_{a,n+m}^{-1} \end{bmatrix} + \begin{bmatrix} 0 & te_{m}^{i\phi} \\ e_{m}^{-i\phi}t^{\dagger} & 0 \end{bmatrix}) \tag{123}$$

If the first matrix is named A and the second B the following structure results after making a series in B.

$$\begin{aligned} TrLog(A+B) &= Tr[Log(A) + \frac{1}{A}B + \frac{1}{2}\frac{1}{A}B\frac{1}{A}B + \frac{1}{3}\frac{1}{A}B\frac{1}{A}B\frac{1}{A}B + \frac{1}{4}\frac{1}{A}B\frac{1}{A}B\frac{1}{A}B\frac{1}{A}B + \mathcal{O}(t^6)] \\ &= Tr[C + \frac{1}{2}\frac{1}{A}B\frac{1}{A}B + \frac{1}{4}\frac{1}{A}B\frac{1}{A}B\frac{1}{A}B\frac{1}{A}B + \mathcal{O}(t^6)] \end{aligned} \tag{124}$$

The first term can be disregarded since it will give an overall constant C. All the odd terms are zero since they only have zero's on the diagonal. The second term will be used to derive the second order kernel.

The partition function is therefore given by:

$$Z = e^{-S_2 - S_4} \tag{125}$$

When considering the second term the following structure is found:

$$S_2 = \frac{1}{2} \sum_{\alpha a} \sum_{nm} (G_{\alpha,n} t e^{i\phi}_m G_{a,n+m} e^{-i\phi}_m t^\dagger + G_{an} e^{-i\phi}_{-m} t^\dagger G_{\alpha n+m} t e^{i\phi}_m) \tag{126}$$

The sum over the Green's function can be made by assuming that the density of state ρ is the same in the leads as in the dot. This relation has been used by many physicists in the past decades and can be found in

$$\sum_i G_i = -\int d\epsilon \rho_{a/\alpha}(\epsilon + \mu) \frac{i\omega_n + \epsilon}{\omega_n^2 + \epsilon^2} \equiv -i\pi \rho_{a/\alpha} \mathrm{Sign}(\omega_n) \tag{127}$$

Here $Sign$ is the Sign function that is either $+$ or $-$. Then the tunneling elements are taken to the front and later included in the conductance parameter g: $tt^\dagger = \mid t \mid^2, \qquad g_t = 2\pi^2 \rho_a \rho_\alpha \mid t \mid^2$. In this approximation it is assumed that $\rho_a \approx \rho_\alpha$. Finally the term with $m = 0$ is added to the first term. This results in an extra factor -1. This then leaves the following summation:

$$S_2 = -\frac{g_t}{2} \sum_{nm} (\mathrm{Sign}(n+m)\mathrm{Sign}(n) - 1) e^{i\phi}_m e^{-i\phi}_m \tag{128}$$

One can solve it as follows: If one assumes $m > 0$ then $\sum_n [S(n)S(n+m) - 1] = {}^{\;\;0\;\; n<-m \;\&\; n>-1}_{-2\; -m<n \;\&\; n<-1}$. Since this logic can also be applied vise versa for $m < 0$ the sum over n can be taken. The sum over n creates m times a factor -2, so:

$$S_2 = -\frac{g_t}{2} \sum_{nm} (\mathrm{Sign}(n+m)\mathrm{Sign}(n) - 1) e^{i\phi}_m e^{-i\phi}_m = \sum_m (-\frac{g_t}{2} 2|m|) e^{i\phi}_m e^{-i\phi}_m \tag{129}$$

By means of the definition of the bosonic frequency this is:

$$S_2 = \sum_m (-g_t \frac{|\nu_m|}{2\pi T}) e^{i\phi}_m e^{-i\phi}_m = g_t \int d\tau \int d\tau' \frac{\sin^2((\phi(\tau) - \phi(\tau')/2))}{\sin^2(\pi T(\tau - \tau'))} \tag{130}$$

Here the relation $\mid m \mid = \frac{|\nu_m|}{2\pi T}$ was used and also the Fourier transform of the action is shown. By this means the part of the action that describes the tunneling phenomena is found to be is proportional to the conductance parameter g. It furthermore shows the characteristics of dissipative damping. Now the fourth order kernel will be calculated by the same method.

9.2. The fourth order correction.

As before the starting point of the calculation is the result after making a series expansion of the action. Below the correction term is given in terms of Green's function's and tunneling elements, as in(126).

$$S_4 = \frac{1}{2} \sum_{\alpha,\alpha',a,a'} \int_{klmn} dt G^{\alpha}_{k-l} e^{i\phi}_l G^{a}_{l-m} e^{-i\phi}_m G^{\alpha'}_{m-n} e^{i\phi}_n G^{a'}_{n-k} e^{-i\phi}_k \tag{131}$$

The Greens function's describe how the system develops between different moments τ_i as follows:

$$G^i_{ab} = \langle \tau_a \mid \frac{1}{\partial_\tau - \mu + \epsilon_i} \mid \tau_b \rangle = \sum_{\omega_a} \frac{e^{i\omega_a(\tau_b - \tau_a)}}{i\omega_a - \mu + \epsilon_i} \tag{132}$$

One can again use the assumption from Altland, Eq. (127) to sum over the Greens function's. The total fourth order contribution is then the following:

$$S_4 = \frac{g_t^2}{2} \sum_{\tilde{\omega_k}\tilde{\omega_l}\tilde{\omega_m}\omega_n} [\mathrm{Sign}(\tilde{\omega_k}) e^{i\phi}_{\tilde{\omega_k}-\tilde{\omega_l}} \mathrm{Sign}(\tilde{\omega_l}) e^{-i\phi}_{\tilde{\omega_l}-\tilde{\omega_m}} \mathrm{Sign}(\tilde{\omega_m}) e^{i\phi}_{\tilde{\omega_m}-\omega_n} \mathrm{Sign}(\omega_n) e^{-i\phi}_{\omega_n-\tilde{\omega_k}}] \tag{133}$$

Here $g_t^2/2 = 2\pi^4 \rho_a^2 \rho_\alpha^2 |t|^4$ as before. This is the starting point of this calculation. As before the goal is to sum over one of the degrees of freedom. This can be done by isolating one of the frequencies. If the frequencies are relabeled by the following re-parameterization the fourth frequency can be isolated:

$$\begin{aligned} \tilde{\omega_k} &= \omega_k + \omega_n \\ \tilde{\omega_l} &= \omega_l + \omega_n \\ \tilde{\omega_m} &= -\omega_m + \omega_l + \omega_n \\ \omega_n &= \omega_n \end{aligned} \tag{134}$$

By this transformation one gets the following structure of the equation:

$$\begin{aligned} S_4 = \frac{g_t^2}{2} \sum_{\omega_k\omega_l\omega_m} & e^{i\phi}_{\omega_k} e^{-i\phi}_{\omega_l} e^{i\phi}_{\omega_m} e^{-i\phi}_{\omega_k-\omega_l+\omega_m} \times \\ & \sum_{\omega_n} [\mathrm{Sign}(\omega_l - \omega_m + \omega_n) \mathrm{Sign}(\omega_k + \omega_n) \mathrm{Sign}(\omega_l + \omega_n) \mathrm{Sign}(\omega_n) - 1] \end{aligned} \tag{135}$$

It is now possible to sum over the final frequency ω_n. However, the equations can be brought in an even shorter form when a second redefinition is made:

$$\begin{aligned} \omega_k &= \omega_1 \\ \omega_l &= \omega_2 \\ \omega_m &= \omega_2 - \omega_3 \\ \omega_n &= \omega_n \end{aligned} \tag{136}$$

This gives the following action.

$$\begin{aligned} S_4 = \frac{g_t^2}{2} \sum_{\omega_1\omega_2\omega_3} & e^{i\phi}_{\omega_1} e^{-i\phi}_{\omega_2} e^{i\phi}_{\omega_2-\omega_3} e^{-i\phi}_{\omega_1-\omega_3} \times \\ & \sum_{\omega_n} [\mathrm{Sign}(\omega_3 + \omega_n) \mathrm{Sign}(\omega_2 + \omega_n) \mathrm{Sign}(\omega_1 + \omega_n) \mathrm{Sign}(\omega_n) - 1] \end{aligned} \tag{137}$$

All representations of the action are equivalent. The most useful equation for the current calculation uses ω_{123n} variables. Later the ω_{klmn} notation will be used for the instanton calculation. Now the summation over n is defined as:

$$N = \sum_{\omega_n} [\text{Sign}(\omega_3 + \omega_n)\text{Sign}(\omega_2 + \omega_n)\text{Sign}(\omega_1 + \omega_n)\text{Sign}(\omega_n) - 1] \tag{138}$$

First a small simplification is made that will help to understand the summation. For now it is assumed that

$$\omega_3 \rangle \omega_2 \rangle \omega_1. \tag{139}$$

With this simplification it is simple to find the regions where the sum is nonzero. It is possible to identify four different cases as can be seen in the first part of Figure 26:

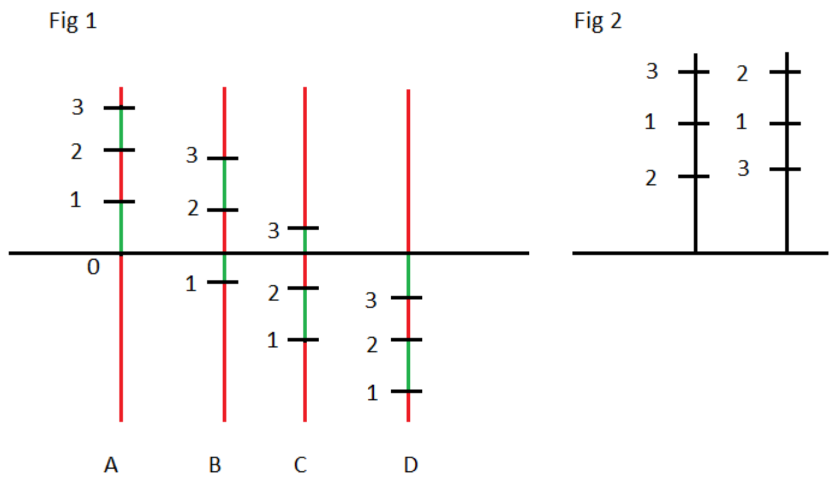

FIG. 26: Here three figures are presented concerning the different order of the frequencies $\omega_1\omega_2\omega_3$.

Each case, labeled A,B,C or D, has a different result for the summation. The different results are given here:

In case A where all three values are positive one finds $2(\omega_3 - \omega_2) + 2\omega_1$.

In case B where the first frequency has a negative value one finds $2(\omega_3 - \omega_2) + 2 \mid \omega_1 \mid$

In case C where just the last frequency is positive one finds $2\omega_3 + 2 \mid \omega_2 - \omega_1 \mid$.

In case D where all frequencies are negative one finds $2 \mid \omega_3 \mid +2(\mid \omega_2 - \omega_1 \mid$.

It is possible to write case D as: $2 \mid \omega_3 \mid -2 \mid \omega_2 \mid +2 \mid \omega_1 \mid$. Then one can see that the cases A , B and C can be written as D because of the trivial fact that the absolute value of a positive frequencies is the frequency itself. This result has been derived under the assumption that the frequencies have values as given in (139). One can easily check that no divergences occur for coinciding values of ω_1, ω_2 and ω_3. The action simplifies in these occasions because of canceling terms as will be shown at the end of this calculation. The order of the frequencies depends on (139). This can be seen in the second part of the figure above. For each frequencies a correction should be made to derive the most general result. In the case where ω_1 is the middle frequency the following result is found:

$$N = 2 \mid \omega_1 \mid -(4 \mid \omega_1 \mid \times \frac{-1}{2}[\text{Sign}(\omega_3 - \omega_1)\text{Sign}(\omega_2 - \omega_1) - 1]) \tag{140}$$

This can be included in our formula's. When all possible orders of frequencies are considered one finds:

$$\begin{aligned} N = {} & 2(\mid \omega_1 \mid + \mid \omega_2 \mid + \mid \omega_3 \mid) \\ & -(4 \mid \omega_1 \mid \times \frac{-1}{2}[\text{Sign}(\omega_3 - \omega_1)\text{Sign}(\omega_2 - \omega_1) - 1]) \\ & -(4 \mid \omega_2 \mid \times \frac{-1}{2}[\text{Sign}(\omega_3 - \omega_2)\text{Sign}(\omega_1 - \omega_2) - 1]) \\ & -(4 \mid \omega_3 \mid \times \frac{-1}{2}[\text{Sign}(\omega_2 - \omega_3)\text{Sign}(\omega_1 - \omega_3) - 1]) \end{aligned} \tag{141}$$

Which simplifies as:

$$\begin{aligned} N = {} & 2(\mid \omega_1 \mid \text{Sign}(\omega_3 - \omega_1)\text{Sign}(\omega_2 - \omega_1) + \mid \omega_2 \mid \text{Sign}(\omega_3 - \omega_2)\text{Sign}(\omega_1 - \omega_2) \\ & + \mid \omega_3 \mid \text{Sign}(\omega_2 - \omega_3)\text{Sign}(\omega_1 - \omega_3)) \\ = {} & -2(\mid \omega_1 \mid \text{Sign}(\omega_3 - \omega_1)\text{Sign}(\omega_1 - \omega_2) + \mid \omega_2 \mid \text{Sign}(\omega_2 - \omega_3)\text{Sign}(\omega_1 - \omega_2) \\ & + \mid \omega_3 \mid \text{Sign}(\omega_2 - \omega_3)\text{Sign}(\omega_3 - \omega_1)) \end{aligned} \tag{142}$$

Here one can see that no problems occur for coinciding values of the frequencies. The summation simplifies when the frequencies have equal values. Then this result is inserted in the fourth order correction to the AES action. First the transformation that was done is presented and then the final result is given:

$$\begin{aligned} \omega_1 &= \omega_k \\ \omega_2 &= \omega_l \\ \omega_3 &= \omega_l - \omega_m \\ \omega_n &= \omega_n \end{aligned} \tag{143}$$

$$S_4 = g_t^2 \sum_{\omega_k \omega_l \omega_m} e^{i\phi}_{\omega_k} e^{-i\phi}_{\omega_l} e^{i\phi}_{\omega_m} e^{-i\phi}_{\omega_k - \omega_l + \omega_m} \times$$
$$\left(\mid \omega_k \mid \frac{(\omega_k - \omega_l)}{\mid \omega_k - \omega_l \mid} \frac{(-\omega_k + \omega_l - \omega_m)}{\mid -\omega_k + \omega_l - \omega_m \mid} - \mid \omega_l \mid \frac{(\omega_k - \omega_l)}{\mid \omega_k - \omega_l \mid} \frac{\omega_m}{\mid \omega_m \mid} + \mid \omega_l - \omega_m \mid \frac{\omega_m}{\mid \omega_m \mid} \frac{(-\omega_k + \omega_l - \omega_m)}{\mid -\omega_k + \omega_l - \omega_m \mid} \right) \tag{144}$$

This term was ignored in the main theory section of this project. It is proportional to g_t^2, the square of the second order coupling constant. Also the action is now written in terms of frequencies for which the factor $\frac{1}{2\pi T}$ compensates. Now this result and its implications can be studied. Therefore this section concludes by checking what happens to the instanton solutions of the action. If this calculation gives a result that makes physically sense it will confirm that all the calculations that were done are correct. Also the effect of the new term will be found as a correction to the second order solution.

9.3. The effect on instantons of S_4.

It is the goal of this section to show that instanton solutions of the path integral still exist when the next order in perturbation theory is taken into account. First a study of the details of the calculation up to second order will be made. It will become apparent that the fourth order term solutions can be calculated with relative ease when the second order calculation is completely understood.

9.4. Instanton solutions to the bare action.

First the bare action and the correction to it as they were calculated are repeated. This allows one to see the early resemblance in the structure of the equations. Below S_2 is given.

$$S_2 = -g_t \frac{T}{\pi} \sum_m |\omega_m| e^{i\phi}_m e^{-i\phi}_m \tag{145}$$

And here the correction that was calculated is repeated:

$$S_4 = g_t^2 (2\pi T)^4 \sum_{\omega_k \omega_l \omega_m} e^{i\phi}_{\omega_k} e^{-i\phi}_{\omega_l} e^{i\phi}_{\omega_m} e^{-i\phi}_{\omega_k - \omega_l + \omega_m} \times$$
$$\left(\mid \omega_k \mid \frac{(\omega_k - \omega_l)}{\mid \omega_k - \omega_l \mid} \frac{(-\omega_k + \omega_l - \omega_m)}{\mid -\omega_k + \omega_l - \omega_m \mid} - \mid \omega_l \mid \frac{(\omega_k - \omega_l)}{\mid \omega_k - \omega_l \mid} \frac{\omega_m}{\mid \omega_m \mid} + \mid \omega_l - \omega_m \mid \frac{\omega_m}{\mid \omega_m \mid} \frac{(-\omega_k + \omega_l - \omega_m)}{\mid -\omega_k + \omega_l - \omega_m \mid} \right) \tag{146}$$

The four summations are defined with a factor : $2\pi T$ and this was omitted up to this point. This results in an extra factor $(2\pi T)^4$ In the action (145) one can substitute the following instanton formula's for the field ϕ:

$$e^{i\phi}(\tau) = \prod_{a=0}^{|W|} \frac{1 - z(\tau)z_a}{z(\tau) - z_a^*} \qquad \text{and} \qquad e^{-i\phi}(\tau) = \prod_{a=0}^{|W|} \frac{1 - z^*(\tau)z_a^*}{z^*(\tau) - z_a} \tag{147}$$

Here $z(\tau) = e^{2\pi i T\tau}$. This allows one to re-parameterize the measure of the integral to a contour integral over the complex unit circle $\mathcal{C}$. From the definition above one can see that $dz/d\tau = -2\pi i T z$ and therefore $(dz)/z = -2\pi i T d\tau$ which yields the following transformation:

$$\int_0^\beta d\tau = -\oint idz(\tau)/(2\pi T z) \tag{148}$$

Combining the instantons with the Fourier Transform of the action results in the following integral:

$$\begin{aligned} S_2 &= \frac{T}{\pi} g_t \sum_{\omega_n} |\omega_n| (\frac{\oint idz(\tau)}{(2\pi T z)}) e^{\omega_n \tau} (\prod_{a=0}^{|W|} \frac{1 - z(\tau)z_a}{z(\tau) - z_a^*}) (\frac{\oint idz^*(\tau')}{(2\pi T z^*)}) e^{-\omega_n \tau'} (\prod_{b=0}^{|W|} \frac{1 - z^*(\tau')z_b^*}{z^*(\tau') - z_b}) \\ &= \frac{T}{\pi} \sum_{\omega_n} |\omega_n| e^{\omega_n \tau} e^{-\omega_n \tau'} (\frac{\oint idz(\tau)}{(2\pi T z)}) (\prod_{a=0}^{|W|} \frac{1 - z(\tau)z_a}{z(\tau) - z_a^*}) (\frac{\oint idz^*(\tau')}{(2\pi T z^*)}) (\prod_{b=0}^{|W|} \frac{1 - z^*(\tau')z_b^*}{z^*(\tau') - z_b}) \end{aligned} \tag{149}$$

The action for $W = 1$ is exactly Eq. (145). It is sufficient to do the calculation for $W = 1$ because both integrals form each others complex conjugate. In the case of multiple terms, due to the product, all new terms will form canceling pairs as will be shown for $W = 1$. Also note that the entire action equals zero if the frequency $\omega_n = 0$. Furthermore, one can write this in terms of the complex variable $z(\tau)$ because $z(\tau)^n = e^{2\pi i n\tau} = e^{i\omega_n \tau}$:

$$S_2 = \frac{T}{\pi} g_t \sum_{\omega_n} \frac{-1}{(2\pi T)^2} |\omega_n| [\oint z^n(\tau) (\frac{1 - z(\tau)z_0}{z(\tau) - z_0^*})][...]^* = \frac{T}{\pi} \sum_{\omega_n} \frac{-1}{(2\pi T)^2} |\omega_n| I_n I_n^* \tag{150}$$

The action is now written as a sum over all frequencies ω_n of the absolute value of ω_n times two independent integrals. The integrals are each others complex conjugate and between them the sign of n differs. If the integer $n \geq 1$ the integral is 0. So n will be negative from this point onwards. Up to this point it has not been specified what the complex parameter z_0 is. For stable instanton solutions the action should be independent of this parameter. Due to the structure of the equations the calculation of the action depends on the location of the parameters z_0. Here these variables are considered to be located outside the complex unit circle in all calculations. The final answer of the calculation does not depend on this choice since the action is invariant under the transformation $z_0 \rightarrow \frac{1}{z_0}$. Therefore, the solutions to the following part of the action will now be derived:

$$\oint dz(\tau) \frac{1}{2\pi T} z^{n-1}(\tau) (\frac{1 - z(\tau)z_0}{z(\tau) - z_0^*}) \tag{151}$$

For that reason the following integral will be considered:

$$I_n = \oint z^{n-1} (\frac{1 - zz_0}{z - z_0^*}) \tag{152}$$

To solve this integral a couple of similar integrals will be considered with specific values for n. First solution is studied for $n = 0$:

$$I_0 = \oint dz \frac{1}{z-\epsilon}\left(\frac{1 - z(\tau)z_0}{z(\tau) - z_0^*}\right) = 2\pi i \frac{1-\epsilon z_0}{\epsilon - z_0^*} = \frac{-2\pi i}{z_0^*} \tag{153}$$

For $n = -1$ one can do the following:

$$I_{-1} = \partial_\epsilon I_0 = 2\pi i\left(\frac{-z_0}{\epsilon - z_o^*} - \frac{1-\epsilon z_0}{(\epsilon - z_0^*)^2}\right) = 2\pi i \frac{(\mid z_0 \mid^2 - 1)}{(z_0^* - \epsilon)^2} = 2\pi i \frac{1 - |z_0|^2}{z_0^{*2}} \tag{154}$$

If the ϵ-derivative is used on (153) n times one will generally get $n! I_n$.

$$I_n = \oint dz(\tau)(z(\tau) - \epsilon)^{n-1}\left(\frac{1 - z(\tau)z_0}{z(\tau) - z_0^*}\right) = 2\pi i(\mid z_0 \mid^2 - 1)(-z_0^*)^{n-1}\frac{n!}{n!} = 2\pi i(\mid z_0 \mid^2 - 1)(z_0^*)^{n-1} \tag{155}$$

This result is stored for a moment to study the other integral. This is the complex conjugate of the last integral with opposite sign for n. First the integral that needs to be calculated is given. Then some simple algebra will be used to make the integrand more similar to the first integral I_n:

$$I_n^* = \oint dz^* z^{*n-1} \frac{1 - z^* z_0^*}{z^* - z_0} \tag{156}$$

Here the notation of $z(\tau) \equiv z$ is used to shorten the formula. This gives the complex conjugate of the answer to the integral I_n.

$$I_n^* = -2\pi i(\mid z_0 \mid^2 - 1)(z_0^{(n-1)}) \tag{157}$$

All what is then left is to multiply both integrals and see what cancels. In the equation below all terms are combined:

$$\begin{aligned} I_n I_n^* &= 4\pi^2(\mid z_0 \mid^2 - 1)(-z_0^*)^{n-1}(\mid z_0 \mid^2 - 1)(-z_0^{(n-1)}) \\ &= 4\pi^2(\mid z_0 \mid^2 - 1)^2(\mid z_0 \mid^2)^{n-1} \end{aligned} \tag{158}$$

Then the sum over all frequencies must be made. First the half way result is presented:

$$\begin{aligned} S_2 = g_t \sum_{\omega_n \leq 0} \frac{T}{\pi}|\omega_n|(4\pi^2)(\mid z_0 \mid^2 - 1)^2(\mid z_0 \mid^2)^{n-1} &= 8\pi T^2 \sum_{n \leq 0} |n|(\mid z_0 \mid^2 - 1)^2(\mid z_0 \mid^2)^{n-1} \\ = 8 g_t \pi T^2 \frac{(\mid z_0 \mid^2 - 1)^2}{(|z_0|^2)} \sum_{n\leq 0} |n|(|z_0|^2)^n \end{aligned} \tag{159}$$

The following structure can be seen:

$$\sum_{n \leq 0} |n|(x)^n = \frac{x}{(x-1)^2} \tag{160}$$

By adding up all factors and constants the following result is found:

$$\begin{aligned} S_2 &= -g_t \frac{T}{\pi}(2\pi i)^2 \frac{(\mid z_0 \mid^2 - 1)^2}{(|z_0|^2-1)^2} \frac{|z_0|^2}{|z_0|^2} 2\pi T \frac{1}{(2\pi T)^2} \\ &= 2g_t \end{aligned} \tag{161}$$

In the final line all canceling terms are dropped. For $W = 1$ one finds that the solution of the action is an integer number. Therefore one can conclude that instanton solutions exist for the action up to second order in perturbation theory.

9.5. Instanton solutions to the action S_4.

With the calculation of the bare action done one can do the calculation for the correction S_4 that was derived before. First the equation that one would like to calculate is written down.

$$\begin{aligned} S_4 =& g_t^2 (2\pi T)^4 \sum_{\omega_k \omega_l \omega_m} e^{i\phi}_{\omega_k} e^{-i\phi}_{\omega_l} e^{i\phi}_{\omega_m} e^{-i\phi}_{\omega_k-\omega_l+\omega_m} \times \\ & \left(\mid \omega_k \mid \frac{(\omega_k-\omega_l)}{\mid \omega_k-\omega_l \mid} \frac{(-\omega_k+\omega_l-\omega_m)}{\mid -\omega_k+\omega_l-\omega_m \mid} - \mid \omega_l \mid \frac{(\omega_k-\omega_l)}{\mid \omega_k-\omega_l \mid} \frac{\omega_m}{\mid \omega_m \mid} + \mid \omega_l - \omega_m \mid \frac{\omega_m}{\mid \omega_m \mid} \frac{(-\omega_k+\omega_l-\omega_m)}{\mid -\omega_k+\omega_l-\omega_m \mid} \right) \end{aligned} \tag{162}$$

Because of the length of the calculation it is comfortable to abbreviate the sum and first solve the four instanton integrals. Therefore the sums argument is defined as a Prefactor $Pf(klm) = \left(\mid \omega_k \mid \frac{(\omega_k-\omega_l)}{|\omega_k-\omega_l|} \frac{(-\omega_k+\omega_l-\omega_m)}{|-\omega_k+\omega_l-\omega_m|} - \mid \omega_l \mid \frac{(\omega_k-\omega_l)}{|\omega_k-\omega_l|} \frac{\omega_m}{|\omega_m|} + \mid \omega_l - \omega_m \mid \frac{\omega_m}{|\omega_m|} \frac{(-\omega_k+\omega_l-\omega_m)}{|-\omega_k+\omega_l-\omega_m|} \right)$. This gives:

$$S_4 = g_t^2 (2\pi T)^4 \sum_{klm} Pf(klm) \times [e^{i\phi}_k e^{-i\phi}_l e^{i\phi}_m e^{-i\phi}_{k-l+m}] \tag{163}$$

Now one can insert the instatons in time representation as was done before:

$$\begin{aligned} S_4 =& g_t^2 (2\pi T)^4 \sum_{klm} Pf(klm) \int_0^\beta d\tau_1 d\tau_2 d\tau_3 d\tau_4 e^{i\omega_k \tau_1 - i\omega_l \tau_2 + i\omega_m \tau_3 - i(\omega_k-\omega_l+\omega_m)\tau_4} \\ & (\prod_{a=1}^{|W|} \frac{1-z(\tau_1)z_a}{z(\tau_1)-z_a^*})(\prod_{b=1}^{|W|} \frac{1-z^*(\tau_2)z_b^*}{z^*(\tau_2)-z_b})(\prod_{c=1}^{|W|} \frac{1-z(\tau_3)z_c}{z(\tau_3)-z_c^*})(\prod_{d=1}^{|W|} \frac{1-z^*(\tau_4)z_d^*}{z^*(\tau_4)-z_d}) \\ \overset{W=1}{\to}& g_t^2 (2\pi T)^4 \sum_{klm} Pf \int_0^\beta d\tau_1 d\tau_2 d\tau_3 d\tau_4 e^{i\omega_k \tau_1 - i\omega_l \tau_2 + i\omega_m \tau_3 - i(\omega_k-\omega_l+\omega_m)\tau_4} \\ & (\frac{1-z(\tau_1)z_1}{z(\tau_1)-z_1^*})(\frac{1-z^*(\tau_2)z_1^*}{z^*(\tau_2)-z_1})(\frac{1-z(\tau_3)z_1}{z(\tau_3)-z_1^*})(\frac{1-z^*(\tau_4)z_1^*}{z^*(\tau_4)-z_1}) \end{aligned} \tag{164}$$

In the last line the winding number W is assumed to be 1 as before because it is not necessary to do this for a general winding number. Then the argument of the four integrals is sorted to simplify the expression:

$$S_4 = g_t^2(2\pi T)^4 \sum_{klm} (\oint \frac{dz(\tau_1)}{2\pi T z(\tau_1)} e^{i\omega_k \tau_1} (\frac{1 - z(\tau_1) z_1}{z(\tau_1) - z_1^*}))(\oint \frac{dz(\tau_2)}{2\pi T z(\tau_2)} e^{-i\omega_l \tau_2} (\frac{1 - z^*(\tau_2) z_1^*}{z^*(\tau_2) - z_1}))\times$$
$$(\oint \frac{dz(\tau_3)}{2\pi T z(\tau_3)} e^{i\omega_m \tau_3} (\frac{1 - z(\tau_3) z_1}{z(\tau_3) - z_1^*}))(\oint \frac{dz(\tau_4)}{2\pi T z(\tau_4)} e^{-i\omega_{k-l+m} \tau_4} (\frac{1 - z^*(\tau_4) z_1^*}{z^*(\tau_4) - z_1})) \times Pf(klm) \tag{165}$$

Notice that the summation is restricted from above by 0 due to Eq (124). This then results in the following restrictions for the variables klm:

$$\begin{aligned} k &\leq 0 \\ l &\leq 0 \\ m &\leq 0 \\ k - l + m &\leq 0 \end{aligned} \tag{166}$$

Now four independent integrals are found. It is possible to solve them if one recalls:

$$I_n = 2\pi i(| z_0 |^2 - 1)(z_0^*)^{n-1} \tag{167}$$

and

$$I_n^* = -2\pi i(| z_0 |^2 - 1)(z_0)^{n-1} \tag{168}$$

If the I integrals are substituted in the action the following expression is found:

$$\begin{aligned} S_4 =& g_t^2(2\pi T)^4 \sum_{\substack{k\leq 0,\\ k+m\leq l\leq 0,\\ m\leq 0}} Pf(klm) \times [I_k I_l^* I_m I_{k-l+m}^*] \frac{1}{(2\pi T)^4} \\ =& g_t^2(2\pi T)^4 \frac{1}{(2\pi T)^4} \sum_{\substack{k\leq 0,\\ k+m\leq l\leq 0,\\ m\leq 0}} Pf(klm) \times [I_k I_l^* I_m I_{k-l+m}^*] \\ =& g_t^2 \sum_{\substack{k\leq 0,\\ k+m\leq l\leq 0,\\ m\leq 0}} Pf(klm) \times \\ & [2\pi i(|z_0|^2 - 1)(z_0^*)^{k-1})(-2\pi i(|z_0|^2 - 1)(z_0)^{l-1})(2\pi i(|z_0|^2 - 1)(z_0^*)^{m-1})(-2\pi i(|z_0|^2 - 1)(z_0)^{k-l+m-1})] \\ =& A \sum_{\substack{k\leq 0,\\ k+m\leq l\leq 0,\\ m\leq 0}} Pf(klm) \times |z_0|^{2k+2m} \end{aligned} \tag{169}$$

Here the same mechanism takes place as was seen in the second order calculation. Also a new prefactor $A = \frac{g_t^2(|z_0|^2-1)^4}{|z_0|^4}$ is introduced for convenience. Now the summation over l can be done. As one can see the sum behaves nicely in the limit where $l \to 0$ and $l \to -\infty$. If one takes the sum over l this last double sum is obtained:

$$
\begin{aligned}
S_4 = {} & A \sum_{\substack{k\leq 0,\\ m\leq 0}} (|z_0|)^{2k+2m} \times \\
& \sum_{k+m\leq l\leq 0} (\mid k \mid \mathrm{Sign}(-m+l-k)\mathrm{Sign}(k-l) + \mid l \mid \mathrm{Sign}(m)\mathrm{Sign}(k-l) + \mid l-m \mid \mathrm{Sign}(m)\mathrm{Sign}(-k+l-m)) \\
& = A \sum_{\substack{k\leq 0,\\ m\leq 0}} (|z_0|)^{2k+2m} \times \sum_{k+m\leq l\leq 0} (\mid k \mid \mathrm{Sign}(k-l) - \mid l \mid \mathrm{Sign}(k-l) - \mid l-m \mid \\
& = A \sum_{\substack{k\leq 0,\\ m\leq 0}} (|z_0|)^{2k+2m} \times (\frac{1}{2} \mid m \mid - \mid k \mid) \\
& = \frac{-1}{2} A \sum_{\substack{k\leq 0,\\ m\leq 0}} (|z_0|)^{2k+2m} \times \mid m \mid
\end{aligned}
\tag{170}
$$

Here the nontrivial summation is that of $Sign(k-l)$ and it gives: $\sum_l Sign(k-l) = \begin{smallmatrix} - & k\leq l\leq 0 \\ + & k+m\leq l\leq 0 \end{smallmatrix}$. This leaves the double summation over k and m which is somewhat similar to the structure that was found in the second order instanton summation. Here the following summations can be used:

$$
\begin{aligned}
s_1 &= \sum_{a\leq 0} (r)^a = \frac{-r}{(1-r)} = \frac{r}{(r-1)} \\
s_2 &= \sum_{a\leq 0} a(r)^a = \frac{-r}{(1-r)^2} \\
s_1 \mid s_2 \mid &= \sum_{\substack{k\leq 0,\\ m\leq 0}} (|z_0|)^{2k+2m} \times \mid m \mid = \frac{-r^2}{(1-r)^3}
\end{aligned}
\tag{171}
$$

By this means the final equations are obtained. First the expressions that were found for the summations are substituted in the action. In the next line the equation gets cleaned up.

$$
\begin{aligned}
S_4 &= -\frac{A}{2} \frac{-|z_0|^4}{(\mid z_0 \mid^2 -1)^3} \\
&= \frac{g_t^2(\mid z_0 \mid^2 -1)^4}{\mid z_0 \mid^4} (\frac{\mid z_0 \mid^4}{(\mid z_0 \mid^2 -1)^3}) \\
&= g_t^2(\mid z_0 \mid^2 -1)
\end{aligned}
\tag{172}
$$

The scale size of the instantons within the unit circle is introduced as $\lambda = \beta(1-|z_0|^2)$. For the anti-instanton the scale size is $\lambda = \beta(1-|z|^{-2})$. The action can be written as:

$$
S_4 = -g_t^2(1- \mid z_0 \mid^2) = g_t^2 \frac{1}{(\frac{\beta}{\lambda}-1)} \tag{173}
$$

It is a satisfying result that one finds from this calculation. The action of the AES-theory was calculated to an extra order in perturbation theory and it does not diverge. Also the correction does not result into an incomprehensible formula. This proves that the AES-theory and the results from Pruisken and Burmistrov are correct up to this new order in perturbation theory. It has been shown that the instanton solutions are temperature independent. One can simply combine the results of the last two paragraphs in a single formula:

$$S_{tot} = S_2 + S_4 = g_t(2 - g_t(1- \mid z_0 \mid^2)) \tag{174}$$

This was the final result of this project and this allows one to make up the conclusions of this investigation.

10. CONCLUSION.

In this section the results of this project are summarized. First a recap is made to the work that was done in this project. Secondly the results of the calculations that were made as a part of this project are reviewed. Finally some future prospects are presented and the implications of this research is highlighted.

10.1. The Single Electron Transistor project.

Starting point of this project was a theoretical study of the Single Electron Transistor. This complex quantum system has attracted much attention recently due to some new discoveries that were made. The SET is a mesoscopic metallic island in close proximity of two other metallic reservoirs. By means of varying the external gate potential and an voltage difference between the reservoirs the quantum mechanical state of the SET can change. Physicists have tried for numerous years to find an electromechanical theory for this system. Prof. A. Pruisken with his colleague Prof I. Burmistrov discovered a new way of analyzing the Hamiltonian of the SET. They described the system by means of a theory of physical observables. A new previously unknown variable was defined, very similar to the external charge of the island. This quasi-charge $q^{'}$ was found to be robustly quantized. In this project the paper of Pruisken and Burmistrov was followed and studied as a starting point for new studies. The theory of physical observables and the renormalization group are used to study the renormalization flow of the SET's coupling constants. These coupling constants are closely related to the charge of and the conductance going through the SET. The theory of physical observables is used to define an effective action and the renormalization group is used to study the temperature dependence of the coupling constants of the system.
The system is first studied without the reservoirs. This system is known as the isolated island. Then the SET was analyzed completely and the renormalization character of the system was derived in two different ways. First the system was studied by means of response theory. This is a theory that has been used for many years to study the SET and similar systems in the quantum regime. With this method perturbation theory is used to reformulate the Hamiltonian of the system. This is done in such a way that the new effective coupling constants can be expressed in correlators of the theory. Secondly an effective action for the system was derived by means of the theory of physical observables. This theory defines new coupling constants by studying the response of the system to small violations of the boundary conditions of the Hamiltonians field ϕ. By means of this new

method two previously unknown quantities were discovered that described the system completely. The new variable q' is found to be robustly quantized. This means that the quasi-conductance $g^{'}$ jumps by a fixed amount on integer values of the quasi-charge $q^{'}$. This is one of the most remarkable features that is shown in the renormalization flow diagram below. To find this diagram the new quasi-variables had to be calculated. It is possible to relate these new variables to the ordinary two-point function of the theory. The last step to take was then to calculate the ordinary correlator and the Matsubara correlator. The Matsubara correlator is a two-point function weighted with the kernel of the theory. It is not possible to give an analytic expression for these correlators that is valid for all values of the conductance parameter g. The physics is very different for a small current then for a large conductance where many electrons tunnel trough the SET. Both correlators for large and small values of g were found. It was then possible to combine the two regimes in a single plot which is given here:

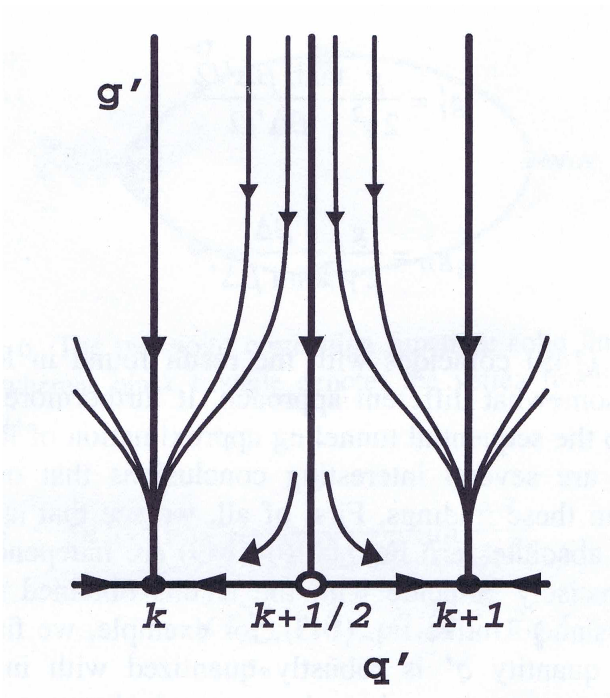

FIG. 27: This diagram gives the unified scaling behavior of the SET. The arrows show scaling for decreasing temperature.

Notice that the coupling constant $q^{'}$ is robustly quantized in this plot. It approaches integer values when the temperature goes to zero. In the upper half of the plot, where the conductance is big, the flow is going down. In this regime the system is dominated by the effect of instanton solutions of the path integral. They are classical solutions of the Hamiltonian for different values of the external charge q'. When g' is small the instantons get overpowered by the effects of the Coulomb blockade and the robust quantization.

10.2. Calculations up to fourth order.

To goal of this project was to provide more detail to the renormalization equations and phase diagram of the SET. That is why, in the second part of this project, an attempt was made to calculate the next order correction to this plot. In this correction the tunneling Hamiltonian is taken into account in more detail. The next order term that is nonzero is the fourth order term. The contribution to the action of this term is given below:

$$\begin{aligned} S_4 =& g_t^2(2\pi T)^4 \sum_{\omega_k\omega_l\omega_m} e^{i\phi}_{\omega_k} e^{-i\phi}_{\omega_l} e^{i\phi}_{\omega_m} e^{-i\phi}_{\omega_k-\omega_l+\omega_m} \times \\ & \left(\mid \omega_k \mid \frac{(\omega_k-\omega_l)}{\mid \omega_k-\omega_l \mid} \frac{(-\omega_k+\omega_l-\omega_m)}{\mid -\omega_k+\omega_l-\omega_m \mid} - \mid \omega_l \mid \frac{(\omega_k-\omega_l)}{\mid \omega_k-\omega_l \mid} \frac{\omega_m}{\mid \omega_m \mid} + \mid \omega_l-\omega_m \mid \frac{\omega_m}{\mid \omega_m \mid} \frac{(-\omega_k+\omega_l-\omega_m)}{\mid -\omega_k+\omega_l-\omega_m \mid} \right) \end{aligned} \tag{175}$$

First the calculation that led to this fourth order correction to the action was presented in full detail. The kernel of the next order correction was found to depend on three different frequencies. Then the effect on the instanton solutions of the theory of this new found action was studied. The calculation of the instanton action up to second and fourth order has been presented. The result of this calculation is repeated here:

$$S_{tot} = S_2 + S_4 = g_t(2 - g_t(1- \mid z_0 \mid^2)) \tag{176}$$

It was found that there are indeed stable minima for the action up to second order. These solutions are scaled down by the fourth order action. It shows that the theory of physical observables based on the AES action is valid up to fourth order perturbations of the tunneling action. The calculations that were made can be used to provide more detail to the renormalization group equations and the phase diagram of the SET.

10.3. Further Prospects.

It was not possible to finish the calculation all the way through to the renormalization β-equations within the limited time of this project. The reason for not finishing the calculations is not relevant for the future of SET studies. It is probable that someone else finishes this calculation. With this correction one obtains a bigger insight in the power and validity of the theory of physical observables. The first steps for analyzing the fourth order have been taken and also a study has been made of the corrections analogue in the Kondo model. From the current status it can be inferred that the corrections to be made will not kill the instanton solutions of the SET action. They will change the critical flow at low values of the conductance, making the renormalization character of the SET more complicated. It is most likely that the revised RG equations will resemble those of the quantum Hall effect more as both theory's have a similar instanton θ-angle in their theory.

Furthermore, all the calculations that were made in the 2010 paper and repeated in this project will some day be put to the test. This day may not even be very far from now as the Aalto University in Helsinki is studying the Single Electron Transistor and is obtaining their first results. Hopefully they will be able to enter the interesting regime of low enough temperatures to confirm the robust quantization that is predicted by this theory. It will not be easy to measure these predictions because the experiment has to trigger the quasi-observables which can not be identified directly from the experimental outset of the system. They may however be triggered and measured by following the methods used to derive them in theory; it may be possible to create a very small periodic distortion to the SET in thermal equilibrium and then study its energy dissipation. Since the dissipation is linked to the effective charge on the island it may show quantization as a function of the frequency of distortion.

There are two possible research interests concerned with the fundamental quantum mechanical nature of the SET. If one takes a broader view the study of the SET has opened up much more possibilities that are not restricted to pure fundamental research on the SET. The SET is so well understood from the theory point of view that it may be a good apparatus for quantum computation. Quantum computation is looking for devices that can make more than one computation at a time. The quantized nature at low temperatures make the SET easy to control and read out. Investigations along this line of thought will continue in the coming years. Also the current calculations on the SET can shed light on other quantum studies. For instance the field of granular metals can benefit from this breakthrough because the new theory that was developed for tunneling can be related too the possible interaction between grains[10, 27–29]. The theory of physical observables has a bright future ahead of itself. It has now shown its use in two very different systems and may be taken up by theorists studying in other fields. It seems an effective way to analyze complex systems that deal with an instanton angle. The research opportunities related with this project are plenty and prosperous.

11. A FINAL WORD.

In this final section of this project it is time to look back on the work that I have done in the past 18 months. Many of the questions and thoughts I had when I started the project have been answered, although not all positive and satisfactory. It is unfortunate to have had so many setbacks in one project as I have had to endure over the last year. The physical research has unfortunately been halted by family loss and important mistakes from my side. When I started the project I intended to push the subject as hard as I could and try to learn as much as possible from this system and the methods to study it. What I regret most is that the learning and working was only done in the latter half of the project. In the first nine months I found out how it is to work on your own on a theoretical physics subject. I did not lack the motivation to study and I still enjoyed the physics concerned with complex quantum systems. What set me back most was the struggle to find a good and efficient way of doing my research. It was the first time in my studies that I had to really work for the physics I was learning about and understand it completely. This was a big lesson and a wake up call but combined with the grief from several funerals it took me until February to start working in a decent pace on the calculations that I was ought to do. I have wondered my entire study how it was to work in theoretical physics, the subject I enjoyed during lectures the most. The hard truth is that working in theory was not a good match for me. In the last months I finally familiarized myself again with programs as *Mathematica* and LaTeX. Skills that I could have used in the first months of this project to learn better and faster. Luckily that knowledge is also very valuable in the field of experiment quantum physics where I look forward to continuing my career. With these program skills and more frequent visits to my supervisor I noticed more and more problems in the calculations I had made. This set me back to almost the beginning of my research and confronted me with how much I had still to do and learn. I am very grateful to my supervisor and friends to both showed patience and pushed me to work. I realize very well how hard it has been for Prof. Pruisken to work with me in all those inefficient and fruitless months. That is why I am very grateful to his honesty and occasional temper that led me into a better working ethos. In the past six months I have been able to understand and learn much more about the reality of physical research and quantum mechanics then in the entire year before that. For all the answers and guidance I received I must thank Prof. Pruisken who taught me everything I needed to learn in this project. I must also thank him for understanding my wish to switch to experimental physics and helping me in many long sessions to finish this project before 2012.

Since these are the last paragraphs I write as a student I must also thank all the wonderful teachers that have given me so much joy and knowledge over the past ±10 years. Starting with my parents whom I am very grateful to for always letting me study the things I enjoyed and pushing me in the right direction when needed. Then first of all Piet Molenaar at the Fons Vitae Lyceum who showed me the joy of learning something about physics. Then at the UvA I was lucky to be taught by good teachers that were extremely open, friendly and damn good in their fields of expertise. The past five years of Bachelor and Master have been extremely joyful and I would like to thank everyone at the UvA, ITFA, NIKHEF, WZI, API and ESC who contributed to that joy of studying.

My last words go to the people that pushed and helped me during this project in a special way. First of all Teuntje Tijssen whom with her love and skills in English helped me tremendously. She also set an example to work hard and pushed me when needed. I am also grateful to my fellow students in the theoretical physics master and especially to Ellen Woerd for all the subjects we enjoyed together and Robert Noest for all the laughs, physics and sports matches we have enjoyed over the last years. Without the help and support of these friends, teachers and family I would not have been able to enjoy studying physics as I did.

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